Cal NERDS Math Vault: Math 1B Solutions

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1 Integrate the following functions

1.1 a)

$$\int x^2 e^x \, dx$$

Solution

Since this integral is composed of two functions, we can integrate using Integration By Parts.

$$u \leftarrow LIATE \rightarrow v \tag{1}$$

$$\int u \, dv = uv - \int v \, du \tag{2}$$

Where L = Logarithmic Functions, I = Inverse Trigonometric Functions, A = Algebraic Functions, T = Trigonometric Functions, and E = Exponential Functions. Thus

$$u = x^2 \Rightarrow du = 2x \, dx$$
$$\int dv = \int e^x \, dx \Rightarrow v = e^x$$

Using the Integration by Parts Formula (2)

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx$$

Using integration by parts one more time for the right integral

$$u = x \Rightarrow du = dx$$
$$\int dv = \int e^x \, dx \Rightarrow v = e^x$$

$$= x^2 e^x - 2[x e^x - \int e^x \, dx]$$

$$\boxed{x^2 e^x - 2x e^x + 2e^x + C}$$

1.2 b)

$$\int sin(x) e^x \, dx$$

Solution

Again, since this integral is composed of two functions, we need to use Integration By Parts. Using LIATE

$$u = sin(x) \Rightarrow du = cos(x) \, dx$$
$$\int dv = \int e^x \Rightarrow v = e^x$$

Then,

$$\int sin(x)e^x = sin(x)e^x - \int e^x cos(x)\,dx$$

Using IBP one more time

$$u = cos(x) \Rightarrow du = -sin(x)\,dx$$
$$\int dv = \int e^x \Rightarrow v = e^x$$

$$= sin(x)e^x - [cos(x)e^x - \int -sin(x)e^x\,dx]$$

$$= sin(x)e^x - cos(x)e^x - \int sin(x)e^x\,dx$$

Since the integral always repeats itself to the original integral, we can use algebra to solve

$$\int sin(x)e^x = sin(x)e^x - cos(x)e^x - \int sin(x)e^x\,dx$$

$$\int sin(x)e^x + \int sin(x)e^x = sin(x)e^x - cos(x)e^x$$

$$2\int sin(x)e^x = sin(x)e^x - cos(x)e^x$$

$$\boxed{= \frac{e^x(sin(x) - cos(x))}{2} + C}$$

1.3 c)

$$\int sin^3 x cos^7 x\,dx$$

Solution

Here we can use u-substitution

$$u = cos(x)$$
$$du = -sin(x)\,dx$$
$$dx = -\frac{1}{sin(x)}\,du$$
$$\int sin^3(x)u^7\,(-\frac{1}{sin(x)}\,du)$$
$$\Rightarrow -\int sin^2(x)u^7\,du$$

We can transform the sine term using the trigonometric identity $sin^2(x) = (1 - cos^2(x))$ to get a cosine term that we can switch to our original u-substitution.

$$-\int (1 - cos^2(x))u^7\,du$$

$$-\int (1 - u^2)u^7\,du$$

$$-\int u^7 - u^9\,du$$

$$-\frac{u^8}{8} + \frac{u^{10}}{10}$$

Substituting back

$$\boxed{= -\frac{cos^8(x)}{8} + \frac{cos^{10}(x)}{10} + C}$$

1.4 d)

$$\int sin^2(x)cos^2(x)\,dx$$

Solution

Since both exponents are even, it is better to use half-angle formulas and trigonometric identities:

$$cos^2(x) = \frac{cos(2x) + 1}{2}$$

$$sin^2(x) = 1 - cos^2(x)$$

Thus

$$\int (1 - cos^2(x))cos^2(x)\,dx$$

$$\int cos^2(x) - cos^4(x)\,dx$$

$$\int \frac{cos(2x) + 1}{2} - (\frac{cos(2x) + 1}{2})^2\,dx$$

At this point is better to breakdown the integral into smaller integrals

$$\frac{1}{2}\int cos(2x)\,dx + \frac{1}{2}\int 1\,dx - \frac{1}{4}\int (cos(2x) + 1)^2\,dx$$

Using u-substitution in the first integrand with u=2x, and expanding the last integrand we get

$$\frac{sin(2x)}{4} + \frac{1}{2}x - \frac{1}{4}\int cos^2(x) + 2cos(2x) + 1\,dx$$

Rearranging terms

$$\frac{sin(2x)}{4} + \frac{1}{2}x - \frac{1}{4}\int 1 + 2cos(2x) + cos^2(x)\,dx$$

We can easily compute the first two terms

$$\frac{sin(2x)}{4} + \frac{1}{2}x - \frac{1}{4}x - \frac{sin(2x)}{4} - \frac{1}{4}\int cos^2(x)\,dx$$

Focusing on the last integrand, we can use half-angle formulas

$$-\frac{1}{4}\int \frac{cos(4x)+1}{2}\,dx$$
$$-\frac{1}{8}\int cos(4x)\,dx - \frac{1}{8}\int 1\,dx$$
$$-\frac{1}{32}\int cos(u)\,du - \frac{1}{8}x$$
$$-\frac{1}{32}sin(4x) - \frac{1}{8}x$$

And now we can eliminate terms and consolidate our answer

$$\frac{sin(2x)}{4} + \frac{1}{2}x - \frac{1}{4}x - \frac{sin(2x)}{4} - \frac{1}{32}sin(4x) - \frac{1}{8}x + C$$
$$\frac{1}{4}x - \frac{1}{8}x - \frac{sin(4x)}{32} + C$$

$$\boxed{= -\frac{sin(4x)}{32} + \frac{1}{8}x + C}$$

1.5 e)

$$\int \sqrt{25 - x^2}\,dx$$

Solution

For this integral we will need to use Trigonometric Substitution. For a radical $\sqrt{a^2 - x^2}$ we can use the substitution $x = asin(\theta)$. Thus

$$\int \sqrt{5^2 - x^2}\,dx$$
$$x = 5sin(\theta)$$
$$dx = 5cos(\theta)d\theta$$

And we can substitute

$$\int \sqrt{25-(5sin(\theta))^2}\, 5cos(\theta)\, d\theta$$

$$\int \sqrt{25-25sin(\theta)}\, 5cos(\theta)\, d\theta$$

$$\int \sqrt{5^2(1-sin^2(\theta))}\, 5cos(\theta)\, d\theta$$

$$\int 5\sqrt{(1-sin^2(\theta))}\, 5cos(\theta)\, d\theta$$

$$25 \int \sqrt{cos^2(\theta)}\, cos(\theta)\, d\theta$$

$$25 \int cos(\theta)cos(\theta)\, d\theta$$

$$25 \int cos^2(\theta)\, d\theta$$

Using half-angle formulas

$$25 \int \frac{cos(2\theta)+1}{2}\, d\theta$$

$$\Rightarrow \frac{25}{4}sin(2\theta) + \frac{25}{2}\theta$$

We can use the double-angle identity: $sin(2\theta) = 2sin(\theta)cos(\theta)$.

$$\frac{25}{2}sin(\theta)cos(\theta) + \frac{25}{2}\theta$$

Since our original susbtitution is based on trigonometry, we can use trigonometric definitions to invert it (see figure 1).

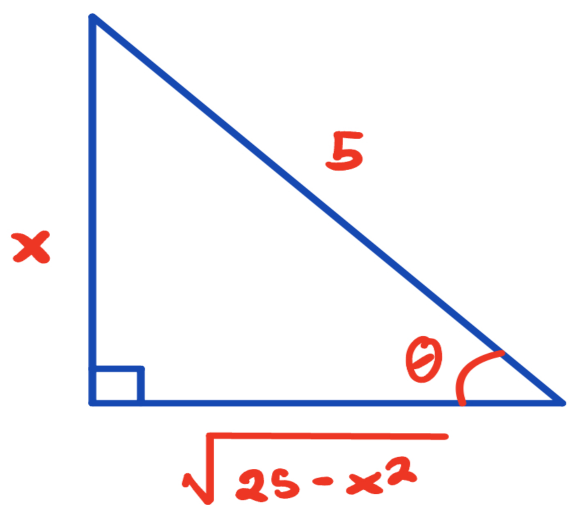

Figure 1: Trigonometric Substitution Triangle

$$x = 5sin(\theta)$$
$$sin(\theta) = \frac{x}{5} = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\theta = sin^{-1}(\frac{x}{5})$$
$$cos(\theta) = \frac{\sqrt{25 - x^2}}{5} = \frac{\text{adjacent}}{hypotenuse}$$

And thus our final solution is

$$\frac{25}{2}(\frac{x}{5})(\frac{\sqrt{25-x^2}}{5}) + \frac{25}{2}sin^{-1}(\frac{x}{5})$$

$$\boxed{= \frac{x\sqrt{25-x^2}}{2} + \frac{25}{2}sin^{-1}(\frac{x}{5}) + C}$$

1.6 f)

$$\int \frac{x^2}{\sqrt{9x^2 - 1}}\, dx$$

Solution

First, let's factor the 9 from the denominator

$$\int \frac{x^2}{\sqrt{9(x^2 - \frac{1}{9})}}\, dx$$
$$\int \frac{x^2}{\sqrt{9}\sqrt{x^2 - \frac{1}{9}}}\, dx$$
$$\int \frac{x^2}{3\sqrt{x^2 - (\frac{1}{3})^2}}\, dx$$

And this integral will require Trigonometric Substitution. For a radical $\sqrt{x^2 - a^2}$ we can use the substitution $x = asec(\theta)$. Thus

$$\int \frac{x^2}{3\sqrt{x^2 - (\frac{1}{3})^2}}\, dx$$
$$x = \frac{1}{3}sec(\theta)$$
$$dx = \frac{1}{3}sec(\theta)tan(\theta)\, d\theta$$

And we can substitute

$$\int \frac{(\frac{1}{3}sec(\theta))^2}{3\sqrt{(\frac{1}{3}sec(\theta))^2 - \frac{1}{9}}} \frac{1}{3} sec(\theta)tan(\theta)\, d\theta$$

$$\int \frac{\frac{1}{9}sec^2(\theta)}{\sqrt{9(\frac{1}{9}sec^2(\theta) - \frac{1}{9})}} \frac{1}{3} sec(\theta)tan(\theta)\, d\theta$$

$$\frac{1}{27}\int \frac{sec^3(\theta)tan(\theta)}{\sqrt{sec^2(\theta)-1}}\, d\theta$$

At this point is always helpful to look at trig identities to check if we can simplify our integrand. Here, we will use the trig identity: $tan(\theta) = \sqrt{sec^2(\theta) - 1}$

$$\frac{1}{27}\int \frac{sec^3(\theta)tan(\theta)}{tan(\theta)}\, d\theta$$

$$\frac{1}{27}\int sec^3(\theta)\, d\theta$$

Because of the odd exponent of the secant, it is helpful to use Integration By Parts. Let's also omit the $\frac{1}{27}$ constant and we'll bring it back later.

$$\int sec(\theta)sec^2(\theta)\, d\theta$$

$$u = sec(\theta) \Rightarrow du = sec(\theta)tan(\theta)\, d\theta$$

$$\int dv = \int sec^2(\theta)\, d\theta \Rightarrow v = tan(\theta)$$

Thus

$$\int sec^3(\theta)\, d\theta = [sec(\theta)tan(\theta) - \int sec(\theta)tan^2(\theta)\, d\theta]$$

Using $tan^2(\theta) = sec^2(\theta) - 1$

$$
\begin{aligned}
&= sec(\theta)tan(\theta) - \int sec(\theta)(sec^2(\theta) - 1)\,d\theta \\
&= sec(\theta)tan(\theta) - \int sec^3(\theta) - sec(\theta)\,d\theta \\
&= sec(\theta)tan(\theta) - \int sec^3(\theta)\,d\theta + \int sec(\theta)\,d\theta
\end{aligned}
$$

$$
\begin{aligned}
\int sec^3(\theta)\,d\theta &= sec(\theta)tan(\theta) + ln|sec(\theta) + tan(\theta)| - \int sec^3(\theta)\,d\theta \\
\int sec^3(\theta)\,d\theta + \int sec^3(\theta)\,d\theta &= sec(\theta)tan(\theta) + ln|sec(\theta) + tan(\theta)| \\
2\int sec^3(\theta)\,d\theta &= sec(\theta)tan(\theta) + ln|sec(\theta) + tan(\theta)| \\
\int sec^3(\theta)\,d\theta &= \frac{sec(\theta)tan(\theta) + ln|sec(\theta) + tan(\theta)|}{2} \\
\int sec^3(\theta)\,d\theta &= \frac{sec(\theta)tan(\theta)}{2} + \frac{ln|sec(\theta) + tan(\theta)|}{2}
\end{aligned}
$$

And now we can bring back the constant

$$
\begin{aligned}
\frac{1}{27}\int sec^3(\theta)\,d\theta &= \frac{1}{27}\left[\frac{sec(\theta)tan(\theta)}{2} + \frac{ln|sec(\theta) + tan(\theta)|}{2}\right] \\
&= \frac{sec(\theta)tan(\theta)}{54} + \frac{ln|sec(\theta) + tan(\theta)|}{54}
\end{aligned}
$$

Now we can use trigonometric definitions to invert out substitution (see figure 5).

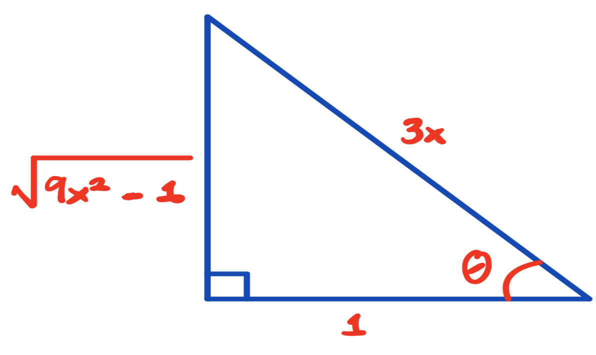

Figure 2: Trigonometric Substitution Triangle

$$x = \frac{1}{3}sec(\theta)$$
$$sec(\theta) = \frac{3x}{1} = \frac{\text{hypotenuse}}{\text{adjacent}}$$
$$tan(\theta) = \frac{\sqrt{9x^2-1}}{1} = \frac{\text{opposite}}{adjacent}$$

$$\Rightarrow \frac{3x\sqrt{9x^2-1}}{54} + \frac{ln|3x+\sqrt{9x^2-1}|}{54}$$

And thus, our final solution is

$$\boxed{= \frac{x\sqrt{9x^2-1}}{18} + \frac{ln|3x+\sqrt{9x^2-1}|}{54} + C}$$

1.7 g)

$$\int \frac{2x^4+3x^3+2x^2+4}{x^7+4x^5+4x^3}\,dx$$

Solution

For this problem we will use Partial Fraction Decomposition. First, let's factorize the denominator as much as we can because this will simplify our process a bit.

$$\int \frac{P(x)}{Q(x)} = \frac{2x^4+3x^3+2x^2+4}{x^3(x^4+4x^2+4)}\,dx$$

The expression x^4+4x^2+4 can easily be simplified to $(x^2+2)^2$, thus

$$\frac{2x^4+3x^3+2x^2+4}{x^3(x^2+2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+2} + \frac{Fx+G}{(x^2+2)^2}$$
$$2x^4+3x^3+2x^2+4 = x^3(x^2+2)^2[\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+2} + \frac{Fx+G}{(x^2+2)^2}]$$
$$2x^4+3x^3+2x^2+4 = Ax^2(x^2+2)^2 + Bx(x^2+2)^2 + C(x^2+2)^2 + (Dx+E)x^3(x^2+2)^2 + (Fx+G)x^3$$

If x=0: $4 = C(2)^2$ which makes $C = 1$. With $C = 1$ we need to expand each part of the expression and rearrange the terms with a common variable. This work leads us to

$$= x^6(A+D) + x^5(B+E) + x^4(4A+1+2D+F) + x^3(4B+2E+G) + x^2(4A+4) + x(4B) + 4$$

And now we can relate the coefficients to the left expression

1. $0x = x(4B) \Rightarrow B = 0$
2. $2x^2 = x^2(A+4) \Rightarrow A = -\frac{1}{2}$
3. $0x^5 = x^5(B+E) \Rightarrow E = 0$

4. $3x^3 = x^3(4B + 2E + G) \Rightarrow G = 3$
5. $0x^6 = x^6(A + D) \Rightarrow D = \frac{1}{2}$
6. $2x^4 = x^4(4(-\frac{1}{2}) + 1 + 2(\frac{1}{2}) + F) \Rightarrow F = 2$

And we can plug back in these constants in our integral breakdown

$$\int \frac{-\frac{1}{2}}{x} + \frac{1}{x^3} + \frac{\frac{1}{2}x}{x^2+2} + \frac{2x+3}{(x^2+2)^2}\,dx$$
$$\int \frac{-\frac{1}{2}}{x}\,dx + \int \frac{1}{x^3}\,dx + \int \frac{\frac{1}{2}x}{x^2+2}\,dx + \int \frac{2x+3}{(x^2+2)^2}\,dx$$

And we can integrate each fraction separately. Our final result is

$$\boxed{= -\frac{1}{2}ln|x| - \frac{1}{2x^2} + \frac{1}{4}ln|x^2+2| - \frac{1}{x^2+2} + \frac{3x}{(x^2+2)^2} + \frac{3\sqrt{2}}{2}tan^{-1}(\frac{x}{\sqrt{2}}) + C}$$

1.8 h)

Estimate the area under the graph in the figure by using (a) the Midpoint Rule and (b) the Trapezoidal Rule each with n = 3, and (c) Simpson's Rule with n = 6.

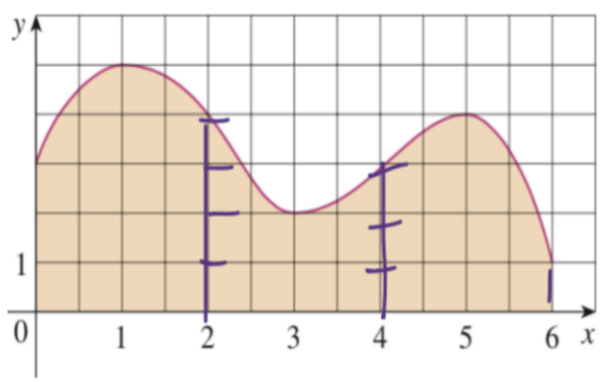

Figure 3: Area under graph

Solution

a) The Midpoint Rule is estimated using the following relation

$$\int_b^a f(x)\,dx \approx \Delta x[f(\bar{x}_1) + ... + f(\bar{x}_i)]$$

Where $\Delta x = \frac{b-a}{n}$ and $\bar{x}_i$ is the midpoint of the ith sub-interval.

$$\Delta x = \frac{b-a}{n} = \frac{6-0}{3} = 2$$

Thus

$$\begin{aligned} Midpoint = M_3 &= \Delta x[f(1) + f(2) + f(3)] \\ &= 2(5 + 2 + 4) \end{aligned}$$

$$\boxed{M_3 \approx 22}$$

b) The Trapezoidal Rule is estimated using the following relation

$$\int_b^a f(x)\,dx \approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + ... + 2f(x_{n-1}) + f(x_n)]$$

Where $\Delta x = \frac{b-a}{n}$ in this case still $\Delta x = 2$. Thus

$$\begin{aligned} Trapezoid = T_3 &= \frac{\Delta x}{2}[f(0) + 2f(2) + 2f(4) + f(6)] \\ &= \frac{2}{2}[3 + 2(4) + 2(3) + 1] \end{aligned}$$

$$\boxed{T_3 \approx 18}$$

c) The Simpson's Rule is estimated using the following relation

$$\int_b^a f(x)\,dx \approx \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + ... + 4f(x_{n-1}) + f(n)]$$

Where $\Delta x = \frac{b-a}{b}$ and n must be even. Thus, $\Delta x = 1$

$$\begin{aligned} Simpson's = S_3 &= \frac{\Delta x}{3}[f(0) + 4f(1) + 2f(2) + 4f(3) + 2f(4) + 4f(5) + f(6)] \\ &= \frac{1}{3}[3 + 4(5) + 2(4) + 4(2) + 2(3) + 4(4) + 1] \end{aligned}$$

$$\boxed{S_3 \approx \frac{20}{3}}$$

2 Find the Arc Length

2.1 a)

$f(x) = x^2 - 3$, $0 \leq x \leq 5$

Solution

We know that if f' is continuous in $[a, b]$, then the length of the curve $y = f(x)$, $a \leq x \leq b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx \tag{3}$$

We take the derivative of the function

$$f'(x) = 2x$$

Thus, the arc length integrand looks like

$$\int_0^5 \sqrt{1 + (2x)^2}\, dx$$

$$\int_0^5 \sqrt{1 + 4x^2}\, dx$$

And this integration will require trigonometric substitution. For a radical $\sqrt{a^2 + x^2}$ we can use the substitution $x = atan(\theta)$. Thus

$$\int \sqrt{1^2 + (2x)^2}\, dx$$

$$2x = tan(\theta)$$

$$x = \frac{1}{2}tan(\theta)$$

$$dx = \frac{1}{2}sec^2(\theta)d\theta$$

And we can substitute

$$\frac{1}{2}\int_0^5 \sqrt{1 + (2 * \frac{1}{2}tan(\theta))^2} sec^2(\theta)\, d\theta$$

$$\frac{1}{2}\int_0^5 \sqrt{1 + tan^2(\theta)} sec^2(\theta)\, d\theta$$

Using the trigonometric identity: $1 + tan^2(\theta) = sec^2(\theta)$

$$\frac{1}{2}\int_0^5 \sqrt{sec^2(\theta)} sec^2(\theta)\, d\theta$$

$$\frac{1}{2}\int_0^5 sec(\theta)sec^2(\theta)\, d\theta$$

$$\frac{1}{2}\int_0^5 sec^3(\theta)\, d\theta$$

Using Integration By Parts with $u = sec(\theta)$ and $\int v = \int sec^2(\theta) = tan(\theta)$ we can integrate

$$\frac{1}{2} * \frac{1}{2}[sec(\theta)tan(\theta) + ln|sec(\theta) + tan(\theta)|]_0^5$$

Now we can use trigonometric definitions to invert out substitution:

$$x = \frac{1}{2}tan(\theta)$$
$$tan(\theta) = \frac{2x}{1} = \frac{\text{opposite}}{\text{adjacent}} = 2x$$
$$sec(\theta) = \frac{1}{cos(\theta)} = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{\sqrt{1+4x^2}}{1} = \sqrt{1+4x^2}$$

And thus, our final solution is

$$\Rightarrow \frac{1}{2} * \frac{1}{2}[\sqrt{1+4x^2}\,2x + ln|\sqrt{1+4x^2} + 2x|]_0^5$$

$$\boxed{= \frac{1}{4}[10\sqrt{101} + ln|\sqrt{101} + 10|]}$$

2.2 b)

$$f(x) = ln(sec(x)),\ 0 \leq x \leq \frac{\pi}{4}$$

Solution

We repeat the same process using equation 3.

$$f'(x) = \frac{sec(x)tan(x)}{sec(x)} = tan(x)$$

And the arc length integrand is

$$\int_0^{\frac{\pi}{4}} \sqrt{1+tan^2(x)}\,dx$$
$$\int_0^{\frac{\pi}{4}} \sqrt{sec^2(x)}\,dx$$
$$\int_0^{\frac{\pi}{4}} sec(x)\,dx$$
$$= ln(|tan(x) + sec(x)|)|_0^{\frac{\pi}{4}}$$

And our final answer is

$$\boxed{= ln(|\sqrt{2} + 1|)}$$

3 Determine whether the following are convergent or divergent

3.1 a)

$$\int_0^1 xln(x)\,dx$$

Solution

The most efficient way to determine if the integral converges or diverges is to take the limit of the integral. Since the function is not continuous at $x = 0$ (that is $(0, 1]$), we can take the limit approaching zero from the left or right. Here I will do it from the left

$$\lim_{t\to 0^+} \int_0^1 xln(x)\,dx$$

We can use Integration By Parts with $u = x$ and $\int v = \int ln(x)$

$$\lim_{t\to 0^+} [\frac{x^2}{2}ln(x) - \frac{1}{2}\int x\,dx]$$
$$\lim_{t\to 0^+} [\frac{x^2}{2}ln(x) - \frac{x^2}{4}]|_t^1$$
$$\lim_{t\to 0^+} [-\frac{1}{4} - \frac{t^2}{2}ln(t) - \frac{t^2}{4}]$$
$$-\frac{1}{4} - \lim_{t\to 0^+} [\frac{t^2}{2}ln(t) + \frac{t^2}{4}]$$

We can separate the terms due to linearity

$$-\frac{1}{4} - [\lim_{t\to 0^+} \frac{t^2}{2}ln(t) + \lim_{t\to 0^+} \frac{t^2}{4}]$$

The second limit goes to zero (by plugging-in zero). We see that the first limit is undefined because $ln(0) = undef$. Thus, we can use L'Hopitals rule

$$\frac{1}{2}\lim_{t\to 0^+} \frac{d}{dt}[t^2 ln(t)]$$

We can rewrite the inside of the limit so it is easier to use L'Hopitals rule

$$\frac{1}{2}\lim_{t\to 0^+}[\frac{\frac{d}{dt}[ln(t)]}{\frac{d}{dt}[\frac{1}{t^2}]}]$$
$$\frac{1}{2}\lim_{t\to 0^+}[\frac{\frac{1}{t}}{-\frac{2}{t^3}}]$$
$$-\frac{1}{2}\lim_{t\to 0^+}[\frac{t^2}{2}]$$
$$-\frac{1}{2}\lim_{t\to 0^+}[\frac{0^2}{2}]$$
$$\Rightarrow 0$$

And the limit of the integral is

$$\boxed{\Rightarrow -\frac{1}{4} = converges}$$

3.2 b)

Sequence

$$a_n = \frac{3}{\sqrt{n^2+4n}-n}$$

Solution

The first step is to rationalize this sequence. Since we want to know if the sequence converges or diverges, we need to set up the limit as n goes to infinity to observe its entire behavior.

$$\lim_{n\to\infty} \frac{3}{\sqrt{n^2+4n}-n} * \frac{\sqrt{n^2+4n}+n}{\sqrt{n^2+4n}+n}$$
$$\lim_{n\to\infty} \frac{3(\sqrt{n^2+4n}+n)}{n^2+4n-n^2}$$
$$\lim_{n\to\infty} \frac{3(\sqrt{n^2+4n}+n)}{4n}$$
$$\lim_{n\to\infty} \frac{3(\sqrt{n^2}\sqrt{1+\frac{4}{n}}+n)}{4n}$$
$$\lim_{n\to\infty} \frac{3(\sqrt{n^2}\sqrt{1+\frac{4}{n}}+n)}{4n}$$
$$\lim_{n\to\infty} \frac{3(n\sqrt{1+\frac{4}{n}}+n)}{4n}$$
$$\lim_{n\to\infty} \frac{3n(\sqrt{1+\frac{4}{n}}+1)}{4n}$$
$$\lim_{n\to\infty} \frac{3(\sqrt{1+\frac{4}{n}}+1)}{4}$$

And we plug in infinity

$$\lim_{n\to\infty} \frac{3(\sqrt{1+\frac{4}{\infty}}+1)}{4}$$
$$\lim_{n\to\infty} \frac{3(\sqrt{1+0}+1)}{4}$$
$$\lim_{n\to\infty} \frac{3(2)}{4}$$

And the answer is

$$\boxed{\Rightarrow \frac{3}{2} = converges}$$

4 Determine whether the following series are divergent or convergent. If convergent, determine the exact value of the series.

4.1 a)

$$\sum_{n=1}^{\infty} \frac{2^{2n+1}}{3^{n-4}}$$

Solution

The Geometric Series theorem states that if $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ...$ converges if and only if, the common ratio $|r| < 1$ (a is a constant). Because we have an n exponent in the denominator and numerator, it is likely we can use the Geometric Series test. We need to rewrite the terms to check if the series is geometric and then check if the common ratio is less than one

$$\sum_{n=1}^{\infty} \frac{2^{2n} * 2^1}{3^n * 3^{-4}}$$

$$\sum_{n=1}^{\infty} \frac{2}{3^{-4}} * \frac{2^{2n}}{3^n}$$

$$\sum_{n=1}^{\infty} \frac{2}{3^{-4}} * \frac{(2^2)^n}{3^n}$$

$$\sum_{n=1}^{\infty} \frac{2}{3^{-4}} * (\frac{4}{3})^n$$

And it does fit the geometric series with a common ratio of $|\frac{4}{3}| > 1$. Thus

$$\boxed{|r| = |\frac{4}{3}| > 1 = \text{Diverges by the Geometric Series Test}}$$

4.2 b)

$$\sum_{n=3}^{\infty} \frac{1}{n^2 + 2n}$$

Solution

Since neither numerator nor denominator have any exponent of n, we cannot perform the Geometric Series test. Since our denominator are n terms raised to exponents, we can try to perform the P-series test. The P-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. Since the denominator has multiple n terms, we first need to perform a Direct Comparison Test. The DCT theorem states that if the $\sum a_n$ and $\sum b_n$ are series with all positive terms (which is true in this case by looking at the function), and is $a_n \leq b_n$ for all n, then: if $\sum b_n$ converges, then $\sum a_n$ converges. We can see that

$$n^2 + 2n > n^2$$

Then, it is easy to see that

$$\frac{1}{n^2 + 2n} < \frac{1}{n^2}$$

Since n^2 converges by the P-series test, then

$$\boxed{\frac{1}{n^2 + 2n} \text{is convergent by the DCT}}$$

4.3 c)

$$a_n = cos(\frac{1}{n})$$

Solution

There are two ways we can answer this problem. One is to simply look at the behavior of a cosine function. Cosine functions oscillate back and forth, which already tells us that the sequence will not converge. Algebraically, we can perform the Divergence Test. If $\lim_{n\to\infty} a_n = 0$ then the sequence converges, if $\lim_{n\to\infty} a_n \neq 0$ then the series diverges.

$$\lim_{n\to\infty} cos(\frac{1}{n})$$
$$\lim_{n\to\infty} cos(\frac{1}{\infty})$$
$$\lim_{n\to\infty} cos(0) = 1 \neq 0$$

Thus

$$\boxed{\Rightarrow cos(\frac{1}{n}) = \text{Diverges by the Divergence Test}}$$

4.4 d)

$$\sum_{n=1}^{\infty} \frac{1}{4n^3 + 17n^2 - 7n - \frac{1}{4}}$$

Solution

We can repeat the same process as b). Since

$$4n^3 + 17n^2 - 7n - \frac{1}{4} > 4n^3$$

We can see that

$$\frac{1}{4n^3 + 17n^2 - 7n - \frac{1}{4}} < \frac{1}{4n^3}$$

Since $\frac{1}{4n^3}$ converges by the P-series test, then

$$\boxed{\frac{1}{4n^3 + 17n^2 - 7n - \frac{1}{4}} \text{is convergent by the DCT}}$$

4.5 e)

Use the Integral Test

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

Solution

The Integral Test theorem states that: suppose $f(n) = a_n$ and $f(x)$ is positive, continuous and decreasing on $[1, \infty]$. Then $\sum_{n=1}^{\infty} a_n$ converges if $\int_1^{\infty} f(x)\,dx$ converges.
Our function does check for being a positive, continuous and decreasing function; thus we can perform the integral test.

$$\int_1^{\infty} \frac{1}{n^3}\,dn$$

$$\lim_{t\to\infty} \int_1^{t} \frac{1}{n^3}\,dn$$

$$-\lim_{t\to\infty} \frac{1}{2n^2}|_1^t$$

$$-\lim_{t\to\infty} [\frac{1}{2t^2} - \frac{1}{2}]$$

$$-\lim_{t\to\infty} [\frac{1}{\infty} - \frac{1}{2}]$$

$$-\lim_{t\to\infty} [0 - \frac{1}{2}]$$

$$= \frac{1}{2}$$

Since the integral converges, so does the sequence.

$$\boxed{\text{Converges by the Integral Test}}$$

4.6 f)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3+5n}$$

Solution

Given that there is an n exponent in the numerator and an n term in the denominator, it is best to use here we the Alternating Series Test. An alternating series is one of the form $\sum(-1)^n b_n$ (or $\sum(-1)^{n+1} b_n$), where $b_n < 0$. If the alternating series $\sum(-1)^n b_n$ satisfies:

1. b_n is decreasing and,
2. $\lim_{n\to\infty} b_n = 0$

Then the series converges. First let's check that the series is decreasing on the interval $[1, \infty)$, for this we can take the first derivate of the function/series. Here we can focus on $b_n = \frac{1}{3+5n}$.

$$f(x) = \frac{1}{3+5x}$$

$$f'(x) = \frac{-5}{(3+5x)^2}$$

It is easy to see that at $x = 1$ the function decreases; as x grows over time, the function decreased over $[1, \infty)$. Now, let's take the limit as n goes to infinity.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{3+5n} = 0$$

So, by the AST, the series converges.

$$\boxed{= \frac{(-1)^{n-1}}{3+5n} = \text{Converges}}$$

4.7 g)

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

Solution

Because of the factorial term in the denominator, it is best to use the Ratio Test here. The Ratio Test theorem states that:

1. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = C < 1$, then $\sum a_n$ is Absolutely Convergent.
2. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = C > 1$ or $= \infty$, then $\sum a_n$ diverges.
3. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$, then the test fails.

Thus

$$\lim_{n\to\infty} |\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}|$$

$$\lim_{n\to\infty} |\frac{(n+1)!}{2^{n+1}} * \frac{n!}{2^n}|$$

Let's rewrite/expand to cancel out terms

$$\lim_{n\to\infty} |\frac{2^n * 2^1}{2^n} * \frac{n(n-1)(n-2)(n-3)\ldots}{(n+1)n(n-1)(n-2)(n-3)\ldots}|$$

$$\lim_{n\to\infty} |2^1 * \frac{1}{(n+1)}|$$

$$\lim_{n\to\infty} |\frac{2}{n+1}|$$

$$\lim_{n\to\infty} |\frac{2}{\infty}| = 0$$

Thus

$$\boxed{\sum_{n=1}^{\infty} \frac{2^n}{n!} \text{ Absolutely Converges by the Ratio Test}}$$

4.8 h)

$$\sum_{n=1}^{\infty}(\frac{n^2+1}{2n^2+1})^n$$

Solution

Since the series is raised to an nth exponent, it is best here to use the Root Test. The Root Test theorem states that:

1. If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = C < 1$, then $\sum a_n$ is Absolutely Convergent.
2. If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = C > 1$ or $= \infty$, then $\sum a_n$ diverges.
3. If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$, then the test fails.

Thus

$$\lim_{n\to\infty}\sqrt[n]{|(\frac{n^2+1}{2n^2+1})|^n}$$
$$\lim_{n\to\infty}\frac{n^2+1}{2n^2+1}$$
$$\lim_{n\to\infty}\frac{n^2(1+\frac{1}{n^2})}{2n^2(1+\frac{1}{2n^2})}$$
$$\lim_{n\to\infty}\frac{n^2(1+\frac{1}{\infty})}{2n^2(1+\frac{1}{\infty})}$$
$$\lim_{n\to\infty}\frac{n^2(1+0}{2n^2(1+0}$$
$$\lim_{n\to\infty}\frac{n^2}{2n^2}$$
$$=\frac{1}{2}<1$$

$$\boxed{\sum_{n=1}^{\infty}(\frac{n^2+1}{2n^2+1})^n \text{ Absolutely Converges by the Root Test}}$$

4.9 i)

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$$

Solution

Since the denominator is raised to a power, we can use the Alternating Series Test. First, let's check that the function b_n is decreasing

$$f(x)=\frac{1}{\sqrt{n}}$$

$$f'(x) = \frac{-1/2}{n^{3/2}}$$

Which is decreasing on $[1, \infty)$. Now, let's check if the series converges or diverges

$$b_n = \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$$

And

$$\lim_{n\to\infty} \frac{1}{\sqrt{\infty + 1}} = 0$$

So

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \text{ Converges by the AST}}$$

5 Find the radius and interval of convergence

5.1 a)

$$\sum_{n=1}^{\infty} (\frac{arctan^n(n)}{2^n} + \frac{1}{n^2})x^n$$

Solution

Let's distribute the x^n term

$$\sum_{n=1}^{\infty} \frac{arctan^n(n)}{2^n} x^n + \frac{x^n}{n^2}$$

By properties of addition

$$\sum_{n=1}^{\infty} \frac{arctan^n(n)}{2^n} x^n + \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

And for the first summation we can perform the root test, while the second summation we can perform the ratio test.

$$\lim_{n\to\infty} \frac{arctan^n(n)}{2^n} x^n$$
$$\lim_{n\to\infty} (\frac{arctan(n) * x}{2})^n$$
$$\lim_{n\to\infty} \sqrt[n]{|(\frac{arctan(n) * x}{2})|^n}$$
$$\lim_{n\to\infty} |\frac{arctan(n) * x}{2}|$$
$$|x| \lim_{n\to\infty} |\frac{arctan(n)}{2}|$$
$$|x| \lim_{n\to\infty} |\frac{arctan(\infty)}{2}|$$
$$|x| * |\frac{\pi/2}{2}|$$
$$|x| * |\frac{\pi}{4}| < 1$$
$$|x * \frac{\pi}{4}| < 1$$
$$|x| < \frac{4}{\pi}$$

Now for the ratio test

$$\lim_{n\to\infty} |\frac{\frac{x^{n+1}}{(n+1)^2}}{\frac{x^n}{n^2}}|$$
$$\lim_{n\to\infty} |\frac{x^{n+1}}{(n+1)^2} * \frac{n^2}{x^n}|$$
$$\lim_{n\to\infty} |\frac{x^n * x^1}{(n+1)^2} * \frac{n^2}{x^n}|$$
$$\lim_{n\to\infty} |\frac{x^1 * n^2}{(n+1)^2}|$$
$$|x| \lim_{n\to\infty} |\frac{n^2}{(n+1)^2}|$$

Applying L'Hopitals

$$|x| \lim_{n\to\infty} |\frac{2n}{2(n+1) * 1}|$$
$$|x| \lim_{n\to\infty} |\frac{n}{(n+1)}|$$
$$|x| * |1| < 1$$
$$|x| < 1$$

So the minimum radius of converge happens between $[-\frac{4}{\pi}, \frac{4}{\pi}], [-1, 1]$. In this case, we know that the series converges in the latter interval. It is still possible that it converges in the former interval; for the former interval, we would need to perform the same limit as n goes to infinity at $x = \frac{-4}{\pi}, \frac{4}{\pi}$ and check if it converges or diverges. In this case, however, the minimum radius of convergence is

$$\boxed{ROC: \ [-1, 1]}$$

5.2 b)

$$f(x) = (\frac{1}{5+x} + \frac{1}{1-3x})$$

Solution

First we need to find a infinite series that represents this function. We know the common power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \text{ for } |x| < 1$$

Now, it is helpful to rewrite the first term of the function by steps. First we factorize the 5 and x in the denominator

$$\frac{1}{5+x} = \frac{1}{5-(-x)} = \frac{1}{5}(\frac{1}{1-(-\frac{x}{5})})$$

And we can substitute $-\frac{x}{5}$ in the power series and expand

$$\sum_{n=0}^{\infty}(-\frac{x}{5})^n$$

$$\sum_{n=0}^{\infty} - \frac{x^n}{5^n}$$

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{5^n}$$

$$\frac{1}{5} * \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{5^n}$$

$$\sum_{n=0}^{\infty}(-1)^n x^n (\frac{1}{5^1 * 5^n})$$

$$\sum_{n=0}^{\infty}(-1)^n x^n (\frac{1}{5})^{n+1}$$

Notice how we transformed the 1 in the numerator into 1^{n+1}. Since 1 raised to any exponent is just one; this allows us to manipulate it to accommodate for the denominator term 5^{n+1}. Now we can do the same for the second term

$$\sum_{n=0}^{\infty}(3x)^n$$
$$\sum_{n=0}^{\infty}3^n x^n$$

We can easily combine the two power series into one

$$\sum_{n=0}^{\infty}(-1)^n x^n (\frac{1}{5})^{n+1} + \sum_{n=0}^{\infty}3^n x^n$$
$$\sum_{n=0}^{\infty}(-1)^n x^n (\frac{1}{5})^{n+1} + 3^n x^n$$
$$\sum_{n=0}^{\infty}(\,(-1)^n (\frac{1}{5})^{n+1} + 3^n)\, x^n$$

And now we have a power series that represents the original function. To find the radius of convergence, we can perform the ratio test separately

$$\sum_{n=0}^{\infty}(\,(-1)^n (\frac{1}{5})^{n+1} x^n + 3^n x^n$$
$$\sum_{n=0}^{\infty}(\,(-1)^n (\frac{1}{5})^{n+1} x^n + \sum_{n=0}^{\infty}3^n x^n$$

$$\lim_{n\to\infty}|\frac{(-1)^{n+1}(1/5)^{n+2}x^{n+1}}{(-1)^n(1/5)^{n+1}x^n}|$$
$$\lim_{n\to\infty}|\frac{(-1)^n(-1)(1/5)^n(1/5)^2x^nx^1}{(-1)^n(1/5)^n(1/5)^1x^n}|$$
$$|\frac{x^nx^1}{x^n}|\lim_{n\to\infty}|\frac{(-1)^n(-1)(1/5)^n(1/5)^2}{(-1)^n(1/5)^n(1/5)^1}|$$
$$|(-1)(1/5)x| < 1$$

$$\lim_{n\to\infty}|\frac{3^{n+1}x^{n+1}}{3^nx^n}|$$
$$\lim_{n\to\infty}|\frac{3^n3^1x^nx^1}{3^nx^n}|$$
$$|\frac{x^nx^1}{x^n}|\lim_{n\to\infty}|\frac{3^n3^1}{3^n}|$$
$$|3x| < 3$$

And the radii of convergence are

$$-5 < x < 5, \text{ and } -1 < x < 1$$

And the minimum radius of converge is

$$\boxed{ROC : [-1, 1]}$$

6 Taylor's Polynomial

6.1 a)

Approximate the function $f(x) = e^{4x} + sin(4x)$ using a Taylor Polynomial $T_n(x)$

Solution

Let's recall the definition that makes up a Taylor series: if $f(x)$ is an infinitely differentiable function on $|x - a| < R$, then its Taylor series is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f"(a)}{2!}(x-a)^2 + \ldots$$

The Taylor series of e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

Using u-substitution, we can just substitute the power of e; that is, let $u = 4x$

$$e^{4x} = \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} = 1 + 4x + \frac{(4x)^2}{2!} + \frac{(4x)^3}{3!} + \ldots + \frac{(4x)^n}{n!}$$

The sine Taylor series is

$$sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \ldots$$

Let $u = 4x$

$$sin(4x) = \sum_{n=0}^{\infty} (-1)^n \frac{(4x)^{2n+1}}{(2n+1)!} = 4x - \frac{(4x)^3}{3!} + \ldots$$

And now, to approximate the Taylor series using the first four terms (that is $T_n(x) = T_4(x)$. We do not bring the $sin(4x)$ Taylor series to $n = 4$ because past $n = 1$ the polynomial has a degree greater than 4, thus changing the Taylor polynomial degree:

$$\begin{aligned}
T_4 &= e^x + sin(x) \\
&= (1 + 4x + \frac{(4x)^2}{2!} + \frac{(4x)^3}{3!}) + \frac{(4x)^4}{4!}) + (4x - \frac{(4x)^3}{3!}) \\
&= 1 + 2(4x) + \frac{(4x)^2}{2!} + \frac{(4x)^3}{3!}) - \frac{(4x)^3}{3!} + \frac{(4x)^4}{4!} \\
&= 1 + 2(4x) + \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!}
\end{aligned}$$

And we can expand the remaining terms

$$\boxed{T_4 \approx 1 + 8x + 8x^2 + \frac{32}{3}x^4}$$

6.2 b)

Centered at $-\frac{1}{4}$, find an N large enough to guarantee the $T_n(x)$ is within 0.1 of $f(x)$ for all x in $[-\frac{1}{2}, 0]$ using the original function in a).

Solution

Here we use Taylor's Inequality: If f^{n+1} is continuous and $|f^{n+1}| \leq M$ between a and x, then:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

First let's find a Taylor series by taking the function's nth derivatives

$$f(x) = e^{4x} + sin(4x)$$
$$f'(x) = 4e^{4x} + 4cos(4x)$$
$$f”(x) = 4^2e^{4x} - 4^2sin(4x)$$
$$f'''(x) = 4^3e^{4x} - 4^3cos(4x)$$
$$f^4(x) = 4^4e^{4x} + 4^4sin(4x)$$
$$...$$
$$f^n(x) = 4^ne^{4x} \pm 4^ncos(4x)$$
$$M = |f^{n+1}(x)| = 4^{n+1}e^{4x} \pm 4^{n+1}cos(4x) \text{ or } sin(4x)$$

No matter what n is, we can see that this is an increasing function, so its maximum on the interval $-\frac{1}{2} < x < 0$ occurs at the right-hand point, $x = 0$. This gives

$$|f^{n+1}(x)| = 4^{n+1}e^{4x} \pm 4^{n+1}cos(4x)$$
$$|f^{n+1}(x)| = 4^{n+1}(e^{4x} \pm cos(4x))$$
$$|f^{n+1}(x)| = 4^{n+1}(e^0 \pm cos(0))$$
$$|f^{n+1}(x)| = 4^{n+1}(1+1)$$
$$|f^{n+1}(x)| = 4^{n+1} * 2$$

And thus

$$|R_n(x)| = |f(x) - T_n| \leq \frac{2 * 4^{n+1}}{(n+1)!}|x + \frac{1}{4}|^{n+1} \text{ on } [-\frac{1}{2}, 0]$$

Similar to the squeeze theorem, on $[-\frac{1}{2}, 0]$ we want to 'sandwich' the $|x + \frac{1}{4}|^{n+1}$.

$$|x + \frac{1}{4}| \leq \frac{1}{4}$$

Since

$$|-\frac{1}{2} + \frac{1}{4}| = |-\frac{1}{4}| \leq \frac{1}{4} \text{ and } |0 + \frac{1}{4}| = |\frac{1}{4}| \leq \frac{1}{4}$$

So

$$|x+\frac{1}{4}|^{n+1} \leq (\frac{1}{4})^{n+1} = \frac{1}{4^{n+1}}$$

Thus

$$\begin{aligned}
|f(x)-T_n| &\leq \frac{2*4^{n+1}}{(n+1)!}|x-\frac{1}{4}|^{n+1} \\
|f(x)-T_n| &\leq \frac{2*4^{n+1}}{(n+1)!}\frac{1}{4^{n+1}} \\
|f(x)-T_n| &\leq \frac{2}{(n+1)!}
\end{aligned}$$

And we want $T_n(x)$ within 0.1 so

$$\begin{aligned}
\frac{2}{(n+1)!} &< 0.1 = \frac{1}{10} \\
\frac{20}{(n+1)!} &< 1 \\
20 &< (n+1)!
\end{aligned}$$

And we test nth points to figure out which one is within 0.1. In this case,

$$\boxed{n \geq 3}$$

7 Power Series Representation

7.1 a)

Find a power series centered at $x=0$ which represents the following function
$f(x) = (x-1)e^{x-1}$.

Solution

We can build the power series strategically. We can start by defining the power series of e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

We multiply times x

$$xe^x = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + ... = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

We multiply times e^{-1}

$$xe^xe^{-1} = xe^{x-1} = \frac{x}{e} + \frac{x^2}{e} + \frac{x^3}{e*2!} + \frac{x^4}{e*3!} + ... = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!*e}$$

Now to define e^{x-1} we can simply factorize an x term

$$e^{x-1} = \frac{1}{e} + \frac{x}{e} + \frac{x^2}{e*2!} + \frac{x^3}{e*3!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!*e}$$

And so we have

$$\Rightarrow xe^{x-1} - e^{x-1}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!*e} - \sum_{n=0}^{\infty} \frac{x^n}{n!*e}$$

And here we are trying to combine the power series, however we have one term with x^{n+1} and the other term with x^n. For this, we can shift the first term by one $n = n - 1$.

$$\Rightarrow \sum_{n=1}^{\infty} \frac{x^{n+1-1}}{(n-1)!*e} - \sum_{n=0}^{\infty} \frac{x^n}{n!*e}$$
$$\Rightarrow \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!*e} - \sum_{n=0}^{\infty} \frac{x^n}{n!*e}$$

And now to make the second power series to be shifted by 1, we can evaluate the first term.

$$\Rightarrow \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!*e} - \sum_{n=0}^{\infty} \frac{x^0}{0!*e}$$
$$\Rightarrow \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!*e} - \frac{1}{e} + \sum_{n=1}^{\infty} \frac{x^n}{n!*e}$$
$$\Rightarrow -\frac{1}{e} + \sum_{n=1}^{\infty} \frac{x^n}{n!*e} + \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!*e}$$
$$\Rightarrow -\frac{1}{e} + \sum_{n=1}^{\infty} [\frac{1}{n!*e} + \frac{1}{(n-1)!*e}]x^n$$

And the power series is

$$\boxed{\Rightarrow -\frac{1}{e} + \sum_{n=1}^{\infty} [\frac{1}{n!} + \frac{1}{(n-1)!}]e^{-1}x^n}$$

7.2 b)

Find a power series centered at $x = 0$ which represents the following function $f(x) = x^2 sin(5x^3)$. Find $f^{29}(0)$ and $f^{30}(0)$.

Solution

Again, this is helpful to do it in steps. First let's start with the sine function

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{(} x)^{2n+1}(2n+1)!$$

Substituting $x = 5x^3$

$$sin(5x^3) = 5x^3 - \frac{(5x^3)^3}{3!} + \frac{(5x^3)^5}{5!} - ... = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1}(x)^{6n+3}}{(2n+1)!}$$

We multiply times x^2

$$x^2 sin(5x^3) = 5x^5 - \frac{x^2(5x^3)^3}{3!} + \frac{x^2(5x^3)^5}{5!} - ... = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1}(x)^{6n+5}}{(2n+1)!}$$

So the power series for the function is

$$\boxed{\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1}(x)^{6n+5}}{(2n+1)!}}$$

To find $f^{29}(x)$ means we need to find the n that makes $x^{6n+5=29}$, which is $n = 4$. For the coefficient of x^k in a power series $\sum c_k x^k$, the k-th derivative at 0 is given by:

$$f^k(0) = k! * c_k$$

Plugging-in $n = 4$

$$\frac{(-1)^4 5^{2*4+1} x^{6*4+5}}{(2*4+1)!} = \frac{5^9 x^{29}}{9!}$$

Thus, the coefficient c_{29} is:

$$c_{29} = \frac{5^9}{9!}$$

The 29th derivative at 0 is then given by:

$$f^{29}(0) = 29! * c_{29} = 29! * \frac{5^9}{9!}$$

For $f^{30}(0)$, $n = \frac{25}{6}$. Since n must be an integer, there is no such n that makes $6n + 5 = 30$. Thus, there is no x^{30} term in the series expansion.

$$\boxed{f^2 9(0) = \frac{5^9 * 29!}{9!}, f^{30}(0) = 0}$$

7.3 c)

$xy" + y' + xy = 0$, $y(0) = 1$, $y'(0) = 1$. What is the pattern for c_{2k} and c_{2k+1}?

Solution

This is a problem of differential equations using series. Here we need to assume that the differential equation solutions take form of MacLaurin Series:

$$y = \sum_{n=0}^{\infty} c_n x^n$$

And we take derivatives accordingly

$$y = \sum_{n=0}^{\infty} c_n x^n$$
$$y' = \sum_{n=1}^{\infty} n * c_n * x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

And now we can multiply by the terms in the given differential equation accordingly

$$xy'' = x * \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}$$
$$xy = x * \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+1}$$

And our differential equation looks like

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} + \sum_{n=1}^{\infty} n * c_n * x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

In order to combine these MacLaurin series let's try to get a common factor of x^n. Pay attention at how we manipulate the nth terms in both the summation and its arguments.

$$\sum_{n+1=2}^{\infty} n(n-1)c_n x^{n-1} + \sum_{n+1=1}^{\infty} n * c_n * x^{n-1} + \sum_{n-1=0}^{\infty} c_n x^{n+1} = 0$$
$$\sum_{n+1=2}^{\infty} (n+1)(n-1+1)c_{n+1} x^{n-1+1} + \sum_{n+1=1}^{\infty} (n+1) * c_{n+1} * x^{n-1+1} + \sum_{n-1=0}^{\infty} c_{n-1} x^{n+1-1} = 0$$
$$\sum_{n=1}^{\infty} (n+1)(n)c_{n+1} x^n + \sum_{n=0}^{\infty} (n+1) * c_{n+1} * x^n + \sum_{n=1}^{\infty} c_{n-1} x^n = 0$$

Now we have a common factor of x^n, however we cannot factorize just yet because the middle summation is centered at $n = 0$. In order to re-center, let's take out the first term (that is, for this summation, plug in $n = 0$), yielding:

$$c_1 + \sum_{n=1}^{\infty} (n+1)(n)c_{n+1} x^n + \sum_{n=1}^{\infty} (n+1)c_{n+1} x^n + \sum_{n=1}^{\infty} c_{n-1} x^n = 0$$
$$c_1 + \sum_{n=1}^{\infty} [(n+1)(n)c_{n+1} + (n+1)c_{n+1} + c_{n-1}]x^n = 0$$

And the differential equation breaks down into

$c_1 = 0$ and,

$$(n+1)(n)c_{n+1} + (n+1)c_{n+1} + c_{n-1} = 0$$

Looking at the second system

$$\begin{aligned}(n+1)[n * c_{n+1} + c_{n+1}] &= -c_{n-1} \\ [n * c_{n+1} + c_{n+1}] &= -\frac{1}{(n+1)}c_{n-1} \\ c_{n+1}(n+1) &= -\frac{1}{(n+1)}c_{n-1} \\ c_{n+1} &= -\frac{1}{(n+1)^2}c_{n-1}\end{aligned}$$

And we check the pattern for c_{2k} values:

1. $c_0 = 1$
2. $c_1 = 0$
3. $c_{1+1} = c_2 = -\frac{1}{2^2}c_0 = -\frac{1}{2^2}$
4. $c_{3+1} = c_4 = -\frac{1}{4^2}c_2 = \frac{1}{2^2 * 4^2}$
5. $c_{5+1} = c_5 = -\frac{1}{6^2}c_4 = -\frac{1}{2^2 * 4^2 * 6^2}$

Giving us

$$...c_{2n} = \frac{(-1)^n}{2^2 * 4^2 * ... * (2n)^2} = \frac{(-1)^n}{4^n (n!)^2}$$

Also, since $c_1 = 0$, we know that $c_3 = c_5 = ... = c_{2n+1} = 0$. Therefore,

$$\boxed{y = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n (n!)^2} x^{2n}}$$

7.4 d)

Write the DE $y" + y' = xy'$ into series form centered at 1.

Solution

Let's rewrite the DE to $y'' - xy' + y' = 0$. Thus

$$\begin{aligned}y &= \sum_{n=0}^{\infty} c_n (x-1)^n \\ y' &= \sum_{n=1}^{\infty} n c_n (x-1)^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) c_n (x-1)^{n-2}\end{aligned}$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n(x-1)^{n-2} - x * \sum_{n=1}^{\infty} nc_n(x-1)^{n-1} + \sum_{n=1}^{\infty} nc_n(x-1)^{n-1}$$

$$\sum_{n=2}^{\infty} n(n-1)c_n(x-1)^{n-2} - [x * \sum_{n=1}^{\infty} nc_n(x-1)^{n-1} - \sum_{n=1}^{\infty} nc_n(x-1)^{n-1}]$$

$$\sum_{n=2}^{\infty} n(n-1)c_n(x-1)^{n-2} - [(x-1) * \sum_{n=1}^{\infty} nc_n(x-1)^{n-1}]$$

$$\sum_{n=2}^{\infty} n(n-1)c_n(x-1)^{n-2} - \sum_{n=1}^{\infty} nc_n(x-1)^{n}$$

$$\sum_{n+2=2}^{\infty} (n+2)(n-1+2)c_{n+2}(x-1)^{n-2+2} - \sum_{n=1}^{\infty} nc_n(x-1)^{n}$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}(x-1)^{n} - \sum_{n=1}^{\infty} nc_n(x-1)^{n}$$

$$2c_2 + \sum_{n=1}^{\infty} (n+2)(n+1)c_{n+2}(x-1)^{n} - \sum_{n=1}^{\infty} nc_n(x-1)^{n}$$

$$\boxed{\Rightarrow 2c_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)c_{n+2} - nc_n](x-1)^{n}}$$

8 Solve the initial value problem for each initial condition $(x+1)^2y' = (1+y)^2$

8.1 a)

$$y(0) = 1$$

Solution

First, let's solve the ODE. Through separation of variables

$$\begin{aligned}
(x+1)^2y' &= (1+y)^2 \\
\frac{dy}{dx} * \frac{1}{(1+y)^2} &= \frac{1}{(x+1)^2} \\
\frac{dy}{(1+y)^2} &= \frac{dx}{(x+1)^2} \\
\int \frac{dy}{(1+y)^2} &= \int \frac{dx}{(x+1)^2} \\
\Rightarrow -\frac{1}{1+y} &= -\frac{1}{1+x} + C
\end{aligned}$$

The initial condition says that at $x = 0$, $y = 1$

$$-\frac{1}{1+1} = -\frac{1}{1+0} + C$$
$$-\frac{1}{2} = -1 + C$$
$$C = \frac{1}{2}$$

Now let's solve for y

$$-\frac{1}{1+y} = -\frac{1}{1+x} + \frac{1}{2}$$
$$-1 = (-\frac{1}{1+x} + \frac{1}{2})(1+y)$$
$$1 + y = \frac{-1}{-(\frac{1}{1+x} - \frac{1}{2})}$$
$$1 + y = \frac{1}{\frac{1}{1+x} - \frac{1}{2}}$$

And after some algebraic manipulation, we get

$$\boxed{y = \frac{2+2x}{2-(1+x)}}$$

8.2 b)

$y(0) = -1$

Solution

We repeat the same process. The initial condition says that at $x = 0$, $y = -1$

$$-\frac{1}{1-1} = -\frac{1}{1+0} + C$$
$$-\frac{1}{0} = -1 + C$$

But $\frac{1}{0}$ is undefined. This means that there is no constant solution that will satisfy $y(0) = -1$.

9 Second Order Non-Homogeneous Differential Equations

9.1 a)

Find the general solution to $y" + 6y' + 9y = e^{-3x} + x$

Solution

Let us start by finding linearly independent solutions

$$\begin{aligned} r^2 + 6r + 9 &= 0 \\ (r+3)^2 &= 0 \end{aligned}$$

Which is a repeated real root, which takes the form of the general solution of the complimentary equation (y_c) $ay'' + by' + cy = 0$:

$$y_c = C_1 e^{-3x} + C_2 x e^{-3x}$$

And now we need to find a particular solution. We can separate both the e^{-3x} and x and find two particular solutions with the undetermined coefficients method. Starting with our first particular solution, notice how we need to multiply times x^2 since xe^{-3x} and just e^{-3x} are repeated terms in the homogeneous solution.

$$\begin{aligned} y_{p_1} &= Ae^{-3x} \\ y_{p_1} &= Ax^2 e^{-3x} \\ y'_{p_1} &= A[2xe^{-3x} - 3x^2 e^{-3x}] \\ y''_{p_1} &= A[2e^{-3x} - 12xe^{-3x} + 9x^2 e^{-3x}] \end{aligned}$$

And we plug in the DE

$$\begin{aligned} &2Ae^{-3x} - 12Axe^{-3x} + 9Ax^2 e^{-3x} + 6A[2xe^{-3x} - 3x^2 e^{-3x}] + 9Ax^2 e^{-3x} = e^{-3x} \\ &2Ae^{-3x} - 12Axe^{-3x} + 9Ax^2 e^{-3x} + 12Axe^{-3x} - 18x^2 e^{-3x} + 9Ax^2 e^{-3x} = e^{-3x} \\ &2Ae^{-3x} = e^{-3x} \Rightarrow A = \frac{1}{2} \end{aligned}$$

For our second particular solution

$$\begin{aligned} y_{p_2} &= Ax + B \\ y'_{p_2} &= A \\ y''_{p_2} &= 0 \end{aligned}$$

$$\begin{aligned} 0 + 6(A) + 9(Ax + B) &= x \\ 6A + 9Ax + 9B &= x \\ 9Ax &= x \Rightarrow A = \frac{1}{9} \\ 6(\frac{1}{9}) + 9B &= 0 \\ \frac{2}{3} &= -9B \Rightarrow B = -\frac{2}{27} \end{aligned}$$

And thus our general solution is

$$\boxed{y_g = C_1 e^{-3x} + C_2 x e^{-3x} + \frac{1}{2}x^2 e^{-3x} + \frac{1}{9}x - \frac{2}{27}}$$

9.2 b)

Find the particular solution to $y" + 6y' + 9y = -3x^{-3x} + x$

Solution

Since it is the same complimentary homogeneous equation as before, we'll focus on the particular solutions.

$$y_{p_1} = (Ax + B)x^2e^{-3x} + C$$
$$y'_{p_1} = Ax^2e^{-3x} + 2x(Ax + B)e^{-3x} - 3(Ax + B)x^2e^{-3x}$$
$$y''_{p_1} = 2Axe^{-3x} - 3Ax^2e^{-3x} + 2(Ax + B)e^{-3x} + 2Axe^{-3x}...$$
$$...-6x(Ax + B)e^{-3} - 3Ax^2e^{-3x} - 6x(Ax + B)e^{-3x} + 9x^2(Ax + B)e^{-3x}$$
$$y''_{p_1} = 4Axe^{-3x} - 6Ax^2e^{-3x} - 12x(Ax + B)e^{-3x} + 2(Ax + B)e^{-3x} + 9x^2(Ax + B)e^{-3x}$$
$$y''_{p_1} = 4Axe^{-3x} - 6Ax^2e^{-3x} - 12Ax^2e^{-3x} - 12Bxe^{-3x} + 2Axe^{-3x} + 2Be^{-3x} + 9Ax^3e^{-3x} + 9Bx^2e^{-3x}$$
$$y''_{p_1} = 9Ax^3e^{-3x} - 12Ax^2e^{-3x} + 6Axe^{-3x} + 9Bx^2e^{-3x} - 12Bxe^{-3x} + 2Be^{-3x}$$

Plugging-in is quite large to include here, but using the same method as a) we get

$$A = -\frac{1}{2}, \ \ B = 0$$

And our second particular solution is the same as a)

$$A = \frac{1}{9}, \ \ B = -\frac{2}{27}$$

And thus the particular solution is

$$\boxed{y_p = -\frac{1}{2}x^2e^{-3x} - \frac{1}{9}x - \frac{2}{27}}$$

9.3 c)

Find the general form of a particular solution to
$y" - 8y' + 16y = 10x^2e^{2x}cos(3x) + 50e^{4x}sin(5x)$

Solution

Using the method of undetermined coefficients when multiple functions multiply looks as follows

$$\boxed{y_p = (Ax^2 + Bx + C)e^{2x}cos(3x) + (Dx^2 + Ex + F)e^{2x}sin(3x) + Ge^{4x}sin(5x) + He^{4x}cos(5x)}$$

If a cosine is involved, we must write the particular solution for both cosine and sine (even if only one is part of the particular solution).

10 Consider the differential equation of the form $y' = F(y)$, where the graph of y' versus y is as follows:

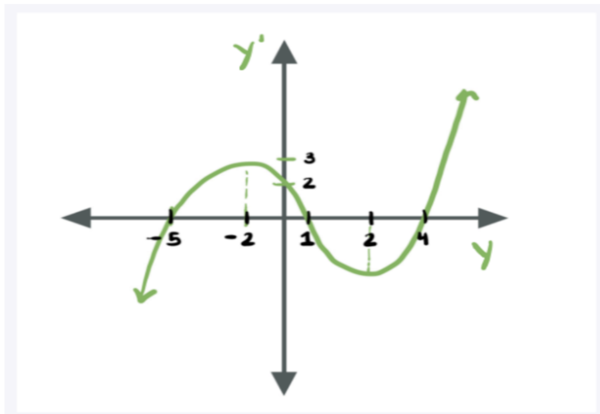

Figure 4: Problem function F(y)

Sketch three solutions to the differential equation one satisfying $y(0) = 6$, another satisfying $y(0) = -6$, the third one satisfies $y(0) = 1$. Determine for what initial conditions $y(0) = y_0$ the solutions have the property that $lim_{x\to\infty} y(x)$ exists and $lim_{x\to-\infty} y(x)$ exists.

Solution

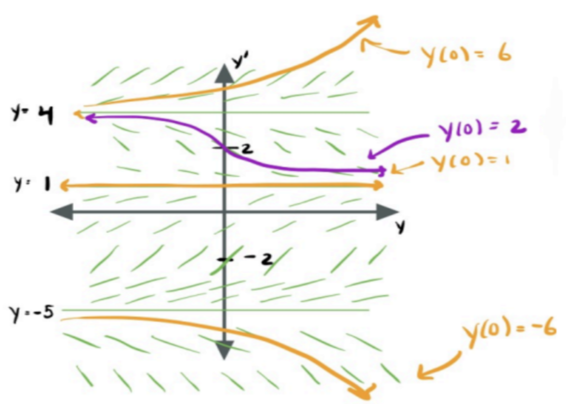

Figure 5: Three solutions satisfying the initial conditions.

11 Bonus Questions

11.1 Find a general solution to the following differential equation $y' = \frac{xln(x)-y}{x}$

Solution

Simplifying a bit

$$y' = ln(x) - \frac{y}{x}$$

$$y' + \frac{y}{x} = ln(x)$$

Here we use the integrating factor

$$\mu(x) = e^{\int \frac{1}{x}} = e^{ln|x|}$$

And we multiply the integrating factor

$$\begin{aligned}
(y * e^{ln|x|})' &= e^{ln|x|} * ln(x) \\
\int (y * e^{ln|x|})' &= \int e^{ln|x|} * ln(x)\, dx \\
y * e^{ln|x|} &= \int e^{ln|x|} * ln(x)\, dx \\
y &= \frac{1}{e^{ln|x|}} \int e^{ln|x|} * ln(x)\, dx \\
y &= \frac{1}{e^{ln|x|}} \int x * ln(x)\, dx \\
y &= \frac{1}{e^{ln|x|}} (\frac{x^2 ln(x)}{2} - \frac{x^2}{4} + C) \\
y &= \frac{1}{x} (\frac{x^2 ln(x)}{2} - \frac{x^2}{4} + C)
\end{aligned}$$

$$\boxed{y = \frac{xln(x)}{2} - \frac{x}{4} + \frac{C}{x}}$$

11.2 Let c_0,c_1,c_2,... and d_0, d_1,d_2,... be sequences of real numbers with the following properties:

- $\sum_{n=0}^{\infty} (-1)^n \frac{c_n 7^n}{3^n}$ **is absolutely convergent.**
- $\sum_{n=0}^{\infty} d_n$ **is conditionally convergent.**

Consider then following power series

$$\sum_{n=0}^{\infty} (n+1)(c_{n+1} + d_{n+1}) x^n$$

For what values of x (if any) is the power series guaranteed to be convergent? For what values of x (if any) is the power series guaranteed to be divergent?

Hint: The sum of a convergent and a divergent series is divergent.

Solution

$$\sum_{n=0}^{\infty}(n+1)(c_{n+1}+d_{n+1})x^n = \sum_{n=0}^{\infty} c_{n+1}nx^n + \sum_{n=0}^{\infty} nd_{n+1}x^n + \sum_{n=0}^{\infty} c_{n+1}x^n + \sum_{n=0}^{\infty} d_{n+1}x^n$$

For $\sum_{n=0}^{\infty} c_{n+1}nx^n$:

$$\lim_{n\to\infty} \frac{(n+1)c_{n+2}x^{n+1}}{nc_{n+1}x^n}$$
$$= \frac{c_{n+2}}{c_{n+1}}x$$

Since $x < 1$, the series is convergent. For $\sum_{n=0}^{\infty} nd_{n+1}x^n$:

$$\lim_{n\to\infty} \frac{(n+1)d_{n+2}x^{n+1}}{nd_{n+1}x^n}$$
$$= \frac{d_{n+2}}{d_{n+1}}x$$

Also since $x < 1$, the series is convergent. For $\sum_{n=0}^{\infty} c_{n+1}x^n$, we see that $\leq c_{n+1}\frac{7^n}{3^n}$, so it is absolutely convergent by CT when $0 \leq x < 1$.
Lastly, for $\sum_{n=0}^{\infty} d_{n+1}x^n$, it is conditionally convergent when $0 \leq x \leq 1$. Thus

$$\boxed{[0,1] : Convergent, \ \ x \notin [0,1] : Divergent}$$

11.3 Determine the convergence or divergence of the following infinite series:

$$\sum_{n=0}^{\infty} \frac{2}{(ln(n)+1)^n}$$

Solution

$$\lim_{n\to\infty} \sqrt[n]{|\frac{2}{(ln(n)+1)^n}|}$$
$$\lim_{n\to\infty} \frac{2^{1/n}}{ln(n)+1} = 0$$

$$\boxed{\text{The series absolutely converges by the Root Test}}$$

11.4 Determine if the following improper integrals are convergent or divergent. Carefully justify your answers.

$$\int_3^\infty \frac{cos(x)+2}{\sqrt[3]{x}\sqrt{x}-2}dx$$

Solution

$$\lim_{t\to\infty}\int_3^t \frac{cos(x)+2}{\sqrt[3]{x}\sqrt{x}-2}dx$$
$$\Rightarrow -1 \le cos(x) \le 1$$
$$\Rightarrow 1 \le cos(x) \le 3$$

Thus

$$\frac{1}{\sqrt[3]{x}\sqrt{x}-2} \le \frac{cos(x)+2}{\sqrt[3]{x}\sqrt{x}-2} \le \frac{3}{\sqrt[3]{x}\sqrt{x}-2}$$

For the left-hand fraction:

$$\lim_{x\to\infty}\left|\frac{\frac{1}{x^{5/6}}}{\frac{1}{\sqrt[3]{x}\sqrt{x}-2}}\right|$$

We see that $\frac{1}{x^{5/6}}$ diverges by p-series since $\frac{5}{6} < 1$.
So,

$$\frac{cos(x)+2}{\sqrt[3]{x}\sqrt{x}-2}$$

is divergent by the LCT, SCT, and p-series.