Cal NERDS Math Vault: Math 1B Calculus Practice Exam

This exam was created in hopes of giving incoming freshmen and transfer students a sense of what college math (2nd semester calculus) at UC Berkeley will be like. This practice exam is to help you prepare for UC Berkeley's Math Courses. If you are unable to complete the exam in the allotted 3 hour time, that is okay. Feel free to continue working on it and check the answers when you are done. If you are using it as a practice test, use the following instructions. Otherwise, your time limit is unlimited!

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Show **all** your work! Take about 3 hours to complete this exam. Try to keep a reasonable pace throughout the exam so you can attempt almost every problem. Skip any problems that you find yourself spending more than 4 minutes on without having made significant progress.

1. Integrate the following functions

   (a)

$$\int x^2 e^x \, dx$$

   (b)

$$\int \sin(x) e^x \, dx$$

(c)

$$\int \sin^3 x \cos^7 x \, dx$$

(d)

$$\int \sin^2 x \cos^2 x \, dx$$

(e)

$$\int \sqrt{25 - x^2}\, dx$$

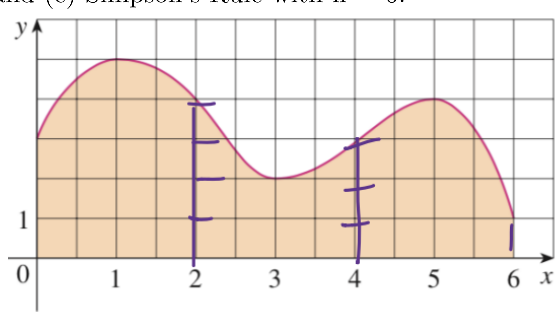
(f)

$$\int \frac{x^2}{\sqrt{9x^2 - 1}}\, dx$$

(g)

$$\int \frac{2x^4 + 3x^3 + 2x^2 + 4}{x^7 + 4x^5 + 4x^3}\, dx$$

(h) Estimate the area under the graph in the figure by using (a) the Midpoint Rule, (b) the Trapezoidal Rule each with n = 3, and (c) Simpson's Rule with n = 6.



2. Find the Arc Length

   (a) $f(x) = x^2 - 3, 0 \leq x \leq 5$

   (b) $f(x) = \ln(\sec x), 0 \leq x \leq \frac{\pi}{4}$

3. Determine whether the following are convergent or divergent.

   (a) If convergent, determine the exact value of the series.

$$\int_0^1 x \ln(x)\, dx$$

   (b) If convergent, determine the exact value of the series.
   Sequence $a_n = \frac{3}{\sqrt{n^2+4n}-n}$

4. Determine whether the following series are convergent or divergent. Try to use different techniques for the various problems.

   (a)

$$\sum_{n=1}^{\infty} \frac{2^{2n+1}}{3^{n-4}}$$

   (b)

$$\sum_{n=3}^{\infty} \frac{1}{n^2 + 2n}$$

(c) $a_n = \cos(\frac{1}{n})$

(d) $\sum_{n=1}^{\infty} \frac{1}{4n^3+17n^2-7n-\frac{1}{4}}$

(e) Use the Integral Test $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(f)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3+5n}$$

(g)

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

(h)

$$\sum_{n=1}^{\infty} (\frac{n^2+1}{2n^2+1})^n$$

(i)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$$

5. Find the radius and interval of convergence.

(a)

$$\sum_{n=1}^{\infty} (\frac{\arctan^n n}{2^n} + \frac{1}{n^2})x^n$$

(b) $f(x) = (\frac{1}{5+x} + \frac{1}{1-3x})$

6. Taylor's Polynomial

   (a) Approximate the function $f(x) = e^{4x} + \sin(4x)$ using a Taylor polynomial $T_n(x)$. Find T_4.

(b) Centered at $-\frac{1}{4}$, find an N large enough to guarantee the $T_n(x)$ is within 0.1 of $f(x)$ for all x in $[-\frac{1}{2}, 0]$.

7. Power Series Representation

(a) Find a power series centered at $x = 0$ which represents the following function: $f(x) = (x-1)e^{x-1}$.

(b) Find a power series centered at $x = 0$ which represents the following function: $f(x) = x^2 \sin(5x^3)$. Find $f^{29}(0)$ and $f^{30}(0)$.

(c) $xy'' + y' + xy = 0$, $y(0) = 1$, $y'(0) = 0$. What is the pattern for c_{2k} and c_{2k+1}?

(d) Find the power series representation $y'' + y' = xy'$ centered at 1

8. Solve the initial value problem for each initial condition $(x+1)^2 y' = (1+y)^2$

   (a) $y(0) = 1$

   (b) $y(0) = -1$

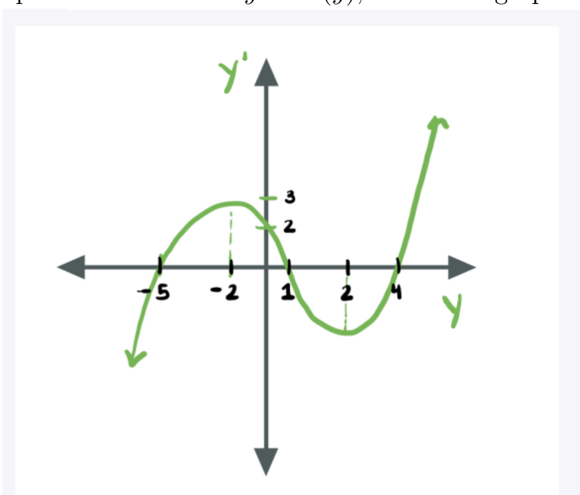
9. Second Order Non-Homogeneous Equations

   (a) Find the general solution to $y'' + 6y' + 9y = e^{-3x} + x$

   (b) Find the partial solution to $y'' + 6y' + 9y = -3xe^{-3x} + x$

(c) Write a trial solution for the particular solution $y'' - 8y' + 16y = 10x^2e^{2x}\cos(3x) + 50e^{4x}\sin(5x)$

10. Consider the differential equation of the form $y' = F(y)$, where the graph of y' versus y is as follows:



Sketch three solutions to the differential equation one satisfying $y(0) = 6$, another satisfying $y(0) = -6$, the third one satisfies $y(0) = 1$. Determine for what initial conditions $y(0) = y_0$ the solutions have the property that $\lim_{x\to\infty} y(x)$ exists and $\lim_{x\to-\infty} y(x)$ exists.

1 Bonus Questions

1. Find a general solution to the following differential equation $y' = \frac{x\ln(x)-y}{x}$.

2. Let $c_0, c_1, c_2, ...$ and $d_0, d_1, d_2...$ be sequences of real numbers with the following property:

   - $\sum_{n=0}^{\infty}(-1)^n \frac{c_n 7^n}{3^n}$ is absolutely convergent.
   - $\sum_{n=0}^{\infty} d_n$ is conditionally convergent.

   Consider the following power series

   $$\sum_{n=0}^{\infty}(n+1)(c_{n+1}+d_{n+1})x^n$$

   For what values of x (if any) is the power series guaranteed to be convergent? For what values of x (if any) is the power series guaranteed to be divergent?
   Hint: The sum of a convergent and a divergent series is divergent.

3. Determine the convergence or divergence of the following infinite series:

$$\sum_{n=1}^{\infty} \frac{2}{(\ln(n)+1)^n}$$

4. Determine if the following improper integrals are convergent or divergent. Carefully justify your answers.

$$\int_3^{\infty} \frac{\cos(x)+2}{\sqrt[3]{x}\sqrt{x}-2}\, dx$$