Cal NERDS Math Vault: Math 1A Solutions

Contact Info: isaacrlin@berkeley.edu and nerds@berkeley.edu

1. Consider the function $f(x) = \sqrt{10-x} + 3$

   (a) Describe in words how to obtain the graph for $f(x)$ from the graph of $\sqrt{x}$.
   Solution.
   Reflect over the y-axis, shift right 10, shift up 3
   OR
   Shift left 10, reflect over the y-axis, shift up 3

   (b) State the domain and range.
   Solution.
   Domain: $(-\infty, 0]$
   Range: $[3, \infty)$

(c) Find $f^{-1}(x)$ and find the domain and range for $f^{-1}(x)$.
Solution.
Swap x and y in the original equation
$y = \sqrt{10-x} + 3 \Rightarrow x = \sqrt{10-y} + 3$
Now solve for y
$x - 3 = \sqrt{10-y}$
$(x-3)^2 = 10 - y$
$\boxed{y = (x-3)^2 - 10}$
The domain of $f^{-1}(x)$ is the range of $f(x)$ and the range of $f^{-1}(x)$ is the domain of $f(x)$ thus:
Domain: $[3, \infty)$
Range: $(-\infty, 0]$

2. Solve for x: $\log_2(x) + \log_2(2x) = 2$
Solution.
$\log_2(2x^2) = 2$
$2x^2 = 2^2$
$x^2 = 2 \Rightarrow x = \pm\sqrt{2}$
To check solutions, plug back into equation
$\log_2(-\sqrt{2}) + \log_2(-2\sqrt{2})$ results in an imaginary so $-\sqrt{2}$ is not a solution
$\log_2(-\sqrt{2}) + \log_2(-2\sqrt{2}) = \frac{1}{2} + \frac{3}{2} = 2$ Thus $\boxed{x = \sqrt{2}}$

3. Evaluate the following limits. If a limit is ∞ or $-\infty$, please say so. Make sure you to show your work and justify your answers.

   (a) $\lim_{x \to 2}(x^2 - 3x + 2)$

   Solution.

   $2^2 - 3(2) + 2 = 4 - 6 + 2 = \boxed{0}$

   (b) $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

   Solution.

   $\lim_{x \to 2} \frac{(x-2)(x+2)}{x-2}$

   $\lim_{x \to 2}(x+2) = \boxed{4}$

(c) $\lim_{x\to 0} \frac{|x-2|-2}{x}$

Solution.

Split into left and right limits

For $x \to 0^+$:

$$\lim_{x\to 1} \frac{|1-2|-2}{1} \Rightarrow \frac{1-2}{1} \Rightarrow -1$$
$$\lim_{x\to 0.5} \frac{|0.5-2|-2}{0.5} \Rightarrow \frac{1.5-2}{0.5} \Rightarrow -1$$
$$\lim_{x\to 0.25} \frac{|0.25-2|-2}{0.25} \Rightarrow \frac{1.25-2}{0.25} \Rightarrow -1$$

For $x \to 0^-$:

$$\lim_{x\to -1} \frac{|-1-2|-2}{-1} \Rightarrow \frac{3-2}{-1} \Rightarrow -1$$
$$\lim_{x\to -0.5} \frac{|-0.5-2|-2}{-0.5} \Rightarrow \frac{2.5-2}{-0.5} \Rightarrow -1$$
$$\lim_{x\to -0.25} \frac{|-0.25-2|-2}{-0.25} \Rightarrow \frac{2.25-2}{-0.25} \Rightarrow -1$$

So $\lim_{x\to 0} \frac{|x-2|-2}{x}$ goes to -1

OR

$|x-2| = \begin{cases} x-2 & x \leq 2 \\ -(x-2) & x > 2 \end{cases}$

$\frac{-(x-2)}{x} = \frac{(2-x)-2}{x} \Rightarrow \frac{-x}{x} = \boxed{-1}$

(d) $\lim_{x\to 0} x \sin(\ln(x^4))$

Solution.

$0 \sin(\ln(0^4)) \Rightarrow 0 \sin(-\infty) \Rightarrow 0 * (-\infty)$

$-1 \leq \sin(\ln(x^4)) \leq 1$

$-|x| \leq \sin(\ln(x^4)) \leq |x|$

$\lim_{x\to 0} |x| \to 0$

$\lim_{x\to 0} -|x| \to 0$

So by the Squeeze Theorem, the limit goes to 0

(e) $\lim_{x\to-\infty}(\sqrt{x^2+3x}-x)$

Solution.

$\sqrt{x^2+3x}-x) \Rightarrow \sqrt{\infty}-(-\infty) \Rightarrow \infty-(-\infty) \Rightarrow \infty$

OR

$$(\sqrt{x^2+3x}-x)*(\frac{\sqrt{x^2+3x}+x}{\sqrt{x^2+3x}+x}) = \frac{x^2+3x-x^2}{\sqrt{x^2+3x}+x} = \frac{3x}{\sqrt{x^2+3x}+x}$$

$$\Rightarrow \frac{3x}{|x|\sqrt{1+\frac{3}{x}}+x} \Rightarrow \frac{3x}{-x\sqrt{1+\frac{3}{x}}+x}$$

$$\Rightarrow \frac{-3}{\sqrt{1+\frac{3}{x}}-1} \Rightarrow \frac{-3}{0^-}$$

So the limit goes to ∞

(f) $\lim_{x\to\infty}(\sqrt{x^2+3x}-x)$

Solution.

$$\lim_{x\to\infty}(\sqrt{x^2+3x}-x) \Rightarrow \infty-\infty$$

$$(\sqrt{x^2+3x}-x)*(\frac{\sqrt{x^2+3x}+x}{\sqrt{x^2+3x}+x}) = \frac{x^2+3x-x^2}{\sqrt{x^2+3x}+x}$$

$$\lim_{x\to\infty}\frac{x^2+3x-x^2}{\sqrt{x^2+3x}+x} = \lim_{x\to\infty}\frac{3x}{\sqrt{x^2+3x}+x} = \lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{3}{x}}+1} = \boxed{\frac{3}{2}}$$

(g) $\lim_{x \to 0} \frac{x^2 e^{3x}}{\tan(x)^2}$
Solution.

$$\lim_{x \to 0} \frac{x^2 e^{3x}}{\tan(x)^2} \Rightarrow \frac{0}{0}$$

Use L'Hopital's:

$$\frac{2xe^{3x} + 3x^2 e^{3x}}{2\tan(x)\sec^2(x)} \to \frac{0}{2(0)(1)} = \frac{0}{0}$$

Use L'Hopital's again:

$$\frac{2e^{3x} + 6xe^{3x} + 6xe^{3x} + 9x^2 e^{3x}}{2\sec^2(x)\sec^2(x)x + 4\tan(x)\sec(x)(\sec(x)\tan(x))} \Rightarrow$$

$$\lim_{x \to 0} \frac{(2 + 12 + 9x^2)e^{3x}}{2\sec^4(x) + 4\tan^2(x)\sec^2(x)} = \frac{(2+0+0)(1)}{2(1) + 4(0)(1)} = \frac{2}{2} = \boxed{1}$$

4. Find the largest value of δ such that is $0 < |x - 2| < \delta$, then $|\frac{1}{x} - \frac{1}{2}| < \frac{1}{4}$. What does this limit represent?
Solution.
First note that $\lim_{x \to 0} \frac{1}{x} = \frac{1}{2}$ so $f(x)$ is between $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ and $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$
so $\frac{1}{x} = \frac{1}{4} \Rightarrow x = 4$ and $\frac{1}{x} = \frac{3}{4} \Rightarrow x = \frac{4}{3}$
Since δ is the distance of x from the limit a, we have 2 choices:
$4 - 2 = 2$ and $2 - \frac{4}{3}$
We will pick the smallest of the 2, since picking 2 would mean we are no longer within the bounds of ϵ, thus $\delta = \frac{2}{3}$

5. Given that $\epsilon = \frac{1}{2}$ find the largest $\delta > 0$ to prove that the given limit $\frac{x}{x-1} = 2$

Solution.

We are given that $\frac{x}{x-1} = 2$ so
$x = 2(x-1) = 2x - 2$
$x = 2$
So $0 < |x-2| < \delta \Rightarrow |\frac{x}{x-1}| < \frac{1}{2}$
The bounds are $2 - \frac{1}{2} = \frac{3}{2}$ and $2 + \frac{1}{2} = \frac{5}{2}$
So $\frac{x}{x-1} = \frac{3}{2} \Rightarrow 3x - 3 = 2x \Rightarrow x = 3$
or
$\frac{x}{x-1} = \frac{5}{2} \Rightarrow 5x - 5 = 2x \Rightarrow x = \frac{5}{3}$
$3 - 2 = 1$ or $2 - \frac{5}{3} = \frac{1}{3}$
so $\epsilon = \frac{1}{3}$

6. Find the first three derivatives of the following functions. Simplify your answers as much as possible. Show all your work.

   (a) $f(x) = x^2 \cos(x)$

   Solution.

   $f'(x) = \boxed{2x\cos(x) - x^2\sin(x)}$

   $f''(x) = 2\cos(x) - 2x\sin(x) - 2x\sin(x) - x^2\cos(x) = \boxed{2\cos(x) - 4x\sin(x) - x^2\cos(x)}$

   $f'''(x) = -2\sin(x) - 4\sin(x) - 4x\cos(x) - 2x\cos(x) + x^2\sin(x)$

$= \boxed{-6\sin(x) - 6x\cos(x) + x^2\sin(x)}$

(b) $f(x) = \frac{x^2-3}{\sqrt{9x-5}}$
Solution.
First use Product Rule:

$$f(x) = \frac{x^2-3}{\sqrt{9x-5}} = (x^2-3)(9x-5)^{-\frac{1}{2}}$$
$$f'(x) = 2x((9x-5)^{-\frac{1}{2}}) - \frac{9}{2}(9x-5)^{-\frac{3}{2}}(x^2-3)$$
$$= \frac{2x}{\sqrt{9x-5}} - \frac{9(x^2-3)}{2(9x-5)^{\frac{3}{2}}} = \boxed{\frac{27x^2-20x+27}{2(9x-5)^{\frac{3}{2}}}}$$

Now use Quotient Rule:

$$f''(x) = \frac{((2)27x-20)(2)(9x-5)^{\frac{3}{2}} - (27x^2-20x+27)(\frac{3}{2})(2)(9x-5)^{\frac{1}{2}}9)}{(2(9x-5)^{\frac{3}{2}})^2}$$
$$= \frac{(108x-40)(9x-5)^{\frac{3}{2}} - (27x^2-20x+27)(27(9x-5)^{\frac{1}{2}})}{4(9x-5)^3}$$
$$= \frac{(108x-40)(9x-5) - (27x^2-20x+27)(27)}{4(9x-5)^{\frac{5}{2}}}$$
$$= \frac{972x^2-900x+200-729x^2+540x-729}{4(9x-5)^{\frac{5}{2}}}$$
$$= \boxed{\frac{243x^2-360x-529}{4(9x-5)^{\frac{5}{2}}}}$$
$$f'''(x) = \frac{((2)(243x)-360)(4)(9x-5)^{\frac{5}{2}} - (243x^2-360x-529)(4)(\frac{5}{2})(9x^2-5)^{\frac{3}{2}}(9)}{(4(9x-5)^{\frac{5}{2}})^2}$$
$$= \frac{((8)(243x)-(4)360)(9x-5)^{\frac{5}{2}} - (243x^2-360x-529)(90)(9x^2-5)^{\frac{3}{2}}}{16(9x-5)^2}$$
$$= \frac{((8)(243x)-(4)360)(9x-5) - (243x^2-360x-529)(90)}{16(9x-5)^2}$$
$$\boxed{\frac{-2187x^2+4860x+27405}{8(9x-5)^{\frac{7}{2}}}}$$

(c) $f(x) = \displaystyle\int_{x^2}^{2} \frac{\cos(t)}{t}\, dt$

Solution.

$f(x) = -\int_2^{x^2} \frac{\cos(t)}{t}\, dt$

$f'(x) = -\frac{\cos(x^2)}{x}(2x) = -2\cos(x^2)(x^{-1})$

$f''(x) = 2\sin(x^2)(2x)(x^{-1}) + (2\cos(x^2)(x^{-1})) = 4\sin(x^2) + \frac{2\cos(x^2)}{x^2}$

$f'''(x) = 8x\cos(x^2) - 4x\sin(x^2)(x^{-2}) - 4\cos(x^2)(x^{-3})$

(d) $f(x) = \frac{\sin(5x)}{x}$
Solution.
Use the Quotient Rule:

$$
\begin{aligned}
f'(x) &= \boxed{\frac{5x\cos(5x) - \sin(5x)}{x^2}} \\
f''(x) &= \frac{x^2(5\cos(5x) - 25x\sin(5x) - 5\cos(5x)) - 2x(5x\cos(5x) - \sin(5x))}{x^4} \\
&= \frac{-25x^2\sin(5x) - 10x\cos(5x) + 2\sin(5x))}{x^3} \\
&\boxed{= \frac{-(25x^2 - 2)\sin(5x) - 10x\cos(5x)}{x^3}} \\
f'''(x) &= \frac{-x^3[50x\sin(5x) + (25x^2 - 2)\cos(5x) * 5 + 10\cos(5x) - 10x\sin(5x) * 5]}{x^6} \ldots \\
&\ldots \frac{+3x^2[(25x^2 - 2)\sin(5x) + 10x\cos(5x)]}{x^6} \\
&= \frac{x[50x\sin(5x) + 125x^2\cos(5x) - 10\cos(5x) + 10\cos(5x) - 50x\sin(5x)]}{x^6} \ldots \\
&\ldots \frac{+3[25x^2\sin(5x) - \sin(5x)10x\cos(5x)]}{x^6} \\
&\boxed{= \frac{(75x^2 - 6)\sin(5x) + (30 - 125x^3)\cos(5x)}{x^4}}
\end{aligned}
$$

(e) $5x + 4xy^2 = 3y + 15$. Specify what is $\frac{dy}{dx}$,

$$
\begin{aligned}
5 + 4y^2 + 8xy\frac{dy}{dx} &= 3\frac{dy}{dx} \Rightarrow 5 + 4y^2 = (3 - 8xy)\frac{dy}{dx} \\
\Rightarrow \frac{dy}{dx} &= \boxed{\frac{5 + 4y^2}{3 - 8xy} = (5 + 4y^2)(3 - 8xy)^{-1}} \\
\frac{d^2y}{dx^2} &= 8y\frac{dy}{dx}(3 - 8xy)^{-1} - (5 + 4y^2)(3 - 8xy)^{-2}(-8y - 8x\frac{dy}{dx}) \\
&= 8y\frac{5 + 4y^2}{3 - 8xy}(3 - 8xy)^{-1} + (5 + 4y^2)(3 - 8xy)^{-2}(8y + 8x)\frac{5 + 4y^2}{3 - 8xy} \\
&\boxed{= 8y\frac{5 + 4y^2}{(3 - 8xy)^2} + (8y + 8x)\frac{(5 + 4y^2)^2}{(3 - 8xy)^3}}
\end{aligned}
$$

(f) $x^2 - \cos(y) = y^3 + 5x - 1$. Specify what is $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$. No need to calculate $\frac{d^3y}{dx^3}$.
Solution.
Use Implicit Differentiation:

$$2x + \sin(y)\frac{dy}{dx} = 3y^2\frac{dy}{dx} + 5$$

$$2x - 5 = (3y^2 - \sin(y))\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \boxed{\frac{2x-5}{3y^2 - \sin(y)}}$$

$$\frac{dy}{dx} = (2x-5)(3y^2 - \sin(y))^{-1}$$

$$\frac{d^2y}{dx^2} = 2(3y^2 - \sin(y))^{-1} + (-1)(2x-5)(3y^2 - \sin(y))^{-2}(6y\frac{dy}{dx} - \cos(y)\frac{dy}{dx})$$

$$= 2(3y^2 - \sin(y))^{-1} - (2x-5)(3y^2 - \sin(y))^{-2}...$$

$$...(6y\frac{2x-5}{3y^2 - \sin(y)} - \cos(y)\frac{2x-5}{3y^2 - \sin(y)})$$

$$\boxed{= 2(3y^2 - \sin(y))^{-1} - (6y - \cos(y))(\frac{(2x-5)^2}{(3y^2 - \sin(y))^3})}$$

(g) $f(x) = \tan^{-1}(x)$
Solution.

$$f'(x) = \boxed{\frac{1}{x^2+1}}$$

$$f''(x) = \frac{-1}{(x^2+1)^2}(2x) = \boxed{\frac{-2x}{(x^2+1)^2}}$$

$$f'''(x) = \frac{-2(x^2+1)^2 - (-2x)(2)(x^2+1)}{(x^2+1)^4} = \frac{-2(x^2+1)^2 - (-8x^2)(x^2+1)}{(x^2+1)^4}$$

$$= \frac{-2(x^2+1) - (-8x^2)}{(x^2+1)^3} = \frac{8x^2 - 2x^2 - 2}{(x^2+1)^3}$$

$$= \boxed{\frac{6x^2 - 2}{(x^2+1)^3}}$$

(h) $f(x) = \sin^{-1}(x)$
Solution.

$$
\begin{aligned}
f'(x) &= \boxed{\frac{1}{\sqrt{1-x^2}}} \\
f''(x) &= -\frac{1}{2}(1-x^2)^{-\frac{3}{2}}(-2x) = \boxed{x(1-x^2)^{-\frac{3}{2}}} \\
f'''(x) &= (1-x^2)^{-\frac{3}{2}} + x(-\frac{3}{2})(1-x^2)^{-\frac{5}{2}}(-2x) = (1-x^2)^{-\frac{3}{2}} + (3x^2)(1-x^2)^{-\frac{5}{2}} \\
&= \frac{1}{(1-x^2)^{\frac{3}{2}}} + \frac{3x^2}{(1-x^2)^{\frac{5}{2}}} = \boxed{\frac{2x^2+1}{(1-x^2)^{\frac{5}{2}}}}
\end{aligned}
$$

7. Suppose you have 100mg of a radioactive substance that decays at a rate of 0.02 mg per day. Find an equation for the amount of substance left after t days.
Solution.
The equation for exponential decay is: $y(t) = y(o)e^{kt}$
Given: $y(0) = 100$ and $y'(0) = -0.02$
$y'(t) = ky(t)$
$\Rightarrow -0.02 = 100k$
$\Rightarrow k = -0.0002$
$\Rightarrow y(t) = 100e^{-0.0002t}$

8. Find the equation of the line tangent to the graph of $f(x) = \frac{1}{x}$ at $x = 8$.
Solution.
When $x - 8$, $y = \frac{1}{8} \Rightarrow f(\frac{1}{x}) = -\frac{1}{x^2} = -\frac{1}{64}$
So $y - \frac{1}{8} = -\frac{1}{64}(x - 8) \Rightarrow y = -\frac{1}{64}x + \frac{1}{8} + \frac{1}{8}$
Thus the equation of the tangent line is $\boxed{y = -\dfrac{1}{64}x + \dfrac{1}{4}}$

9. The sum of two non-negative numbers is 10

   (a) What is the minimum sum of their squares?
   Solution.
   We are given the constraint: $x + y = 10$
   The objective is: $x^2 + y^2 = s$
   $y = 10 - x$
   $\Rightarrow x^2 + (10 - x)^2 = s$
   $\Rightarrow x^2 + 100 - 10x - 10x + x^2 = s$
   $\Rightarrow 2x^2 - 20x + 100 = s$
   $\Rightarrow 4x - 20 = 0 \Rightarrow x = 5$
   Now plug this value of x back into the original constraint
   $5 + y = 10 \Rightarrow y = 5$
   So the minimum sum of their squares is $5^2 + 5^2 = 25 + 25 = 50$

(b) What are the two numbers that minimize the sum of their squares?
Solution.
x=5
y=5

(c) What is the maximum sum of their squares?
Solution.
Consider the end points:
x=10 and y = 0
OR
x=0 and y=10
In either case, the maximum sum would be $10^2 + 0 = 0 + 10^2 = 100$

10. Approximate the area under $f(x) = \sqrt{x+1}$ with four equal subintervals for x-values on the interval $0 \leq x \leq 8$. Use the left Riemann sum, right Riemann sum, midpoint Riemann sum, and trapezoidal sum.

Solution.

x	0	2	4	6	8
y	1	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{7}$	3

Left Riemann Sum: $2(f(0) + f(2) + f(4) + f(6) + f(8)) = \boxed{2(1 + \sqrt{3} + \sqrt{5} + \sqrt{7})}$

Right Riemann Sum: $2(f(0) + f(2) + f(4) + f(6)) = \boxed{2(\sqrt{3} + \sqrt{5} + \sqrt{7} + 3)}$

Midpoint Riemann Sum: $2(f(1) + f(3) + f(5) + f(7)) = \boxed{2(\sqrt{2} + 2 + \sqrt{6} + \sqrt{8})}$

Trapezoidal Sum: $\frac{8-0}{4} = 2 \Rightarrow \frac{2}{2} = 1$

$1(f(0) + 2f(2) + 2f(4) + 2f(6) + f(8)) = \boxed{1 + 2\sqrt{3} + 2\sqrt{5} + 2\sqrt{7} + 3}$

11. Integrate the following functions.

(a) $f(x) = \sec^2 x$

Solution.

$\int \sec^2(x)dx = \boxed{\tan x + c}$

(b) $g(x) = 11 + \frac{5}{x}$

Solution.

$\int 11 + \frac{5}{x} dx = \boxed{11x + 5\ln|x| + c}$

(c) $h(x) = \frac{1}{2\sqrt{x}} - e^x + \cos(x)$

Solution.

$\int \frac{1}{2\sqrt{x}} - e^x + \cos(x) dx = \int \frac{1}{2\sqrt{x}} dx - \int e^x dx + \int \cos(x) dx$

$\boxed{= \sqrt{x} - e^x + \sin(x) + c}$

(d) $p(x) = \frac{7e^{x^5}\cos(7x) - 5x^4 e^{x^5}\sin(7x)}{e^{2x^5}}$

Solution.

$p(x) = 7e^{-x^5}\cos(7x) - 5x^4 e^{-x^5}\sin(7x)$

$\Rightarrow \int 7e^{-x^5}\cos(7x) - 5x^4 e^{-x^5}\sin(7x)dx$

$u = e^{-x^5}\sin(7x)$

$\Rightarrow du = 7e^{-x^5}\cos(7x) - 5x^4 e^{-x^5}\sin(7x)dx$

$\Rightarrow \int du = u + c$

$\boxed{\Rightarrow e^{-x^5}\sin(7x) + c}$

(e) The acceleration of a particle moving along the x-axis is given by $a(t) = e^t - 10t$ for time $0 \leq t \leq 14$. The particle's initial velocity is 13 and its initial position is -3.

i. What is the particle's velocity function? Use it to find $v(2)$. What is the particle's position function? Use it to find its position at time $t = 1$.

Solution.

To find the velocity function, take the integral of acceleration:

$v(t) = \int a(t) = (e^t - 10t)dt = e^t - 5t^2 + c$

$v(0) = 13 = e^0 - 5(0)^2 + c \Rightarrow c = 12$

$\Rightarrow v(t) = e^t - 5t^2 + 12$

$\Rightarrow v(2) = e^2 - 5(2)^2 + 12 = \boxed{e^2 - 8}$

Now to find the position function, take the integral of velocity:

$p(t) = \int(e^t - 5t^2 + 12)dx = e^t - \frac{5}{3}t^3 + 12x + c$

$p(0) = -3 = e^0 - \frac{5}{3}(0)^3 + 12(0) + c \Rightarrow c = -4$

$\Rightarrow p(t) = e^t - \frac{5}{3}t^3 + 12x - 4$

$\Rightarrow p(1) = e^1 - \frac{5}{3}(1)^3 + 12(1) - 4 = \boxed{e - \frac{19}{3}}$

ii. What was the particle moving to the left or the right at time $t = 2$? Explain your reasoning.
Solution.
In part i, we found that at $t = 2$, $v(2) = e^2 - 8$
$e^2 < 8$, so the particle is moving towards the $\boxed{left}$

iii. Was the particle moving toward or away from the origin at time $t = 3$? Give a reason for your answer?
Solution.
$p(3) = e^3 - \frac{5}{3}t(3)^3 + 12(3)x - 4 \Rightarrow p(3) = e^3 - 13$
$e^3 < 13$
$v(3) = e^3 - 5(3)^2 + 12 \Rightarrow v(3) = e^3 - 33$
$e^3 < 33$
Thus the particle is moving $\boxed{toward}$ the origin

(f) $f(x) = x^2 \cos(5x^3)$

Solution.

Use a U-sub: let $u = 5x^3 \Rightarrow du = 15x^2 dx \Rightarrow \frac{du}{15x^2} = dx$

$\int x^2 \cos(u) \frac{du}{15x^2} = \frac{1}{15} \int \cos(u) du$

$= \frac{1}{15} \sin(u) + c$

$$\boxed{= \frac{1}{15} \sin(5x^3) + c}$$

(g) $g(x) = \frac{1}{x^2} \sec(\frac{3}{x}) \tan(\frac{3}{x})$

Solution.

$\int \frac{1}{x^2} \sec(\frac{3}{x}) \tan(\frac{3}{x}) dx$

Use a U-sub: $u = \frac{3}{x} \Rightarrow du = -\frac{3}{x^2}$

$-\frac{1}{3} \int \sec(u) \tan(u) du$

$-\frac{1}{3} \sec(u) + c$

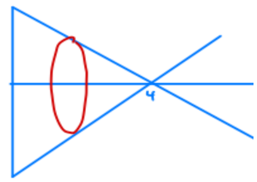
$$\boxed{-\frac{1}{3} \sec(\frac{3}{x}) + c}$$

12. Find the average value of $f(x) = \sqrt{1-x^2}$ on the interval $[-1, 1]$
Solution.
To find the average value, use the following equation:
$f_{ave} = \frac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$
$\Rightarrow \frac{1}{1-(-1)}\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{1}{2}\int_{-1}^{1} \sqrt{1-x^2}\,dx$
This function represents the upper semicircle of the unit circle centered about the origin
so $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{\pi}{2}$
Average Value $= (\frac{1}{2})(\frac{\pi}{2}) = \boxed{\frac{\pi}{4}}$

13. Consider the two functions $f(x) = x^2$ and $g(x) = 8 - x^2$. Find the area bound by these two curves.
Solution.
First, find the bounds for the integral:
$f(x) = g(x) \Rightarrow x^2 = 8 - x^2 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$
$\int_{-2}^{2}(8-x^2)-(x^2)dx = \int_{-2}^{2}(8-2x^2)dx = [8(2) - \frac{16}{3}] - [8(-2) - 2\frac{(-2)^3}{3}]$
$= [16 - \frac{16}{3}] - [-16 - 2\frac{-8}{3}] = 32 - \frac{32}{3} = \boxed{\frac{64}{3}}$

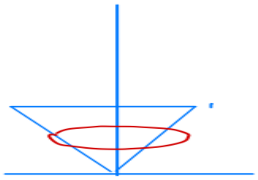
14. Set-up and integral for its volume, and then evaluate the integral. Sketching out the graph might help. Use disks.

(a) The region R enclosed by $y = 12 - 3x$ and $y = 0$ on $[0, 4]$ is revolved about the x-axis.
Solution.



$r = 12 - 3x - 0 = 12 - 3x$
$A = \pi r^2 = \pi(12 - 3x)^2$
We integrate the area to find the volume:
$V = \int_0^4 \pi(12 - 3x)^2 dx = \int_0^4 \pi(9x^2 - 72x + 144)dx$
$= \pi(3x^3 - 36x^2 + 144x)|_0^4$
$= \pi(192 - 585 + 585) = \boxed{192\pi}$

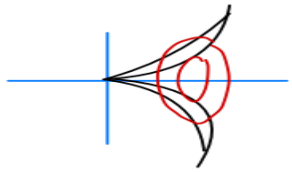
(b) The region R is enclosed by $y = \frac{x}{6}$, $y = 1$, $x = 0$ is revolved about the y-axis.
Solution.



$r = 6y - 0 = 6y$
$A = \pi r^2 = \pi(6y)^2$
We integrate the area to find the volume:
$V = \int_0^1 \pi(6y)^2 dy = \pi \int_0^1 36y^2 dy$
$= \pi(12(1)^3 - 12(0)^3) = \boxed{12\pi}$

15. Set-up and integral for its volume, and then evaluate the integral. Sketching out the graph might help. Use washers.

    (a) The region R enclosed by $y = x^2$ and $y = x^5$ is revolved about the x-axis.
    Solution.

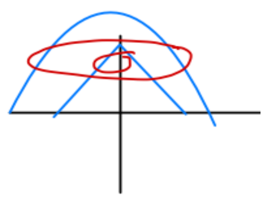


$r = x^2 - 0$ and $R = x^5 - 0$

$V = \pi \int_0^1 (r^2 - R^2)dx$

$= \pi \int_0^1 (x^4 - x^{10})dx = \pi(\frac{1}{5}x^5 - \frac{1}{11}x^{11})|_0^1 = \pi(\frac{1}{5} - \frac{1}{11}) = \boxed{\frac{6\pi}{55}}$

    (b) The region R is enclosed by $y = 9 - x^2$ and $y = 9 - 3x$ is revolved about the y-axis.
    Solution.



Find the integral limits:

$9 - x^2 = 9 - 3x \Rightarrow x^2 = 3x \Rightarrow x = 0, 3$

Now find the equations in terms of y:

$y = 9 - x^2 \Rightarrow x = \sqrt{9 - y}$

$y = 9 - 3x \Rightarrow x = 3 - \frac{1}{3}y$

Finally take the integral:

$V = \pi \int_0^1 (r^2 - R^2)dx$

$V = \pi \int_0^1 ((\sqrt{9 - y})^2 - (3 - \frac{1}{3}y)^2)dx = \pi \int_0^1 9 - y - (9 - 2y + \frac{1}{9}y^2)dx$

$= \pi \int_0^1 (y - \frac{1}{9}y^2)dx = \pi(\frac{1}{2}y^2 - \frac{1}{27}y^3)|_0^3 = \boxed{\frac{7}{2}\pi}$

1 Bonus Questions

1. Use Newton's method with initial approximation $x_1 = 0$ and two-steps to approximate a solution of the equation $\ln(x+1) = 1/2$. Show all your work.
   The Newton's method is the following: We wish to solve an equation of the form $f(x) = 0$, so the roots of the equation correspond to the x-intercepts of the graph of f. We start with a first approximation x_1. Consider a tangent line L (see figure below), labeled x_2. The idea behind Newton's method is that the tangent line is close to the curve and so its x-intercept, x_2, is close to the x-intercept of the curve (namely, the root r that we are seeking). Because the tangent line is a line, we can easily find its x-intercept.

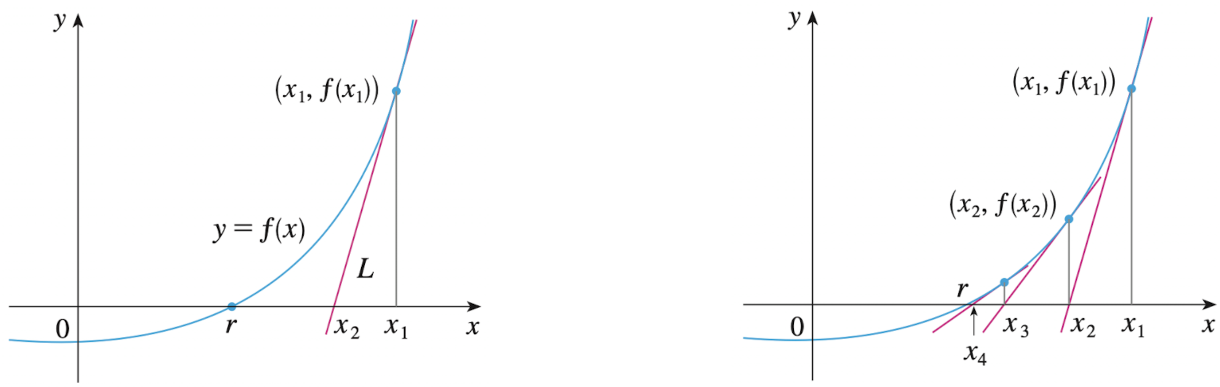


Figure 1: Multiple approximations x_1 to x_{n+1} in Newton's method

The formula for Newton's method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

If the numbers x_n become closer and closer to r as n become large, then we say that the sequence converges to r and we write:

$$\lim_{n \to \infty} x_n = r$$

Thus,

$$f(x) = ln(x+1) - \frac{1}{2}$$
$$f'(x) = \frac{1}{x+1}$$

We insert the first approximation x_1

$$f(x_1) = ln(0+1) - \frac{1}{2}$$
$$f(x_1) = ln(1) - \frac{1}{2}$$
$$f(x_1) = -\frac{1}{2}$$

Therefore,

$$x_2 = x_1 - \frac{f(x)}{f'(x)}$$
$$x_2 = 0 - \frac{-1/2}{1} = \frac{1}{2}$$
$$f(x_2) = ln(1/2+1) - \frac{1}{2}$$
$$f(x_2) = ln(3/2) - \frac{1}{2}$$
$$f'(x_2) = (1+1/2)^{-1} - \frac{2}{3}$$

Finally, for our last step

$$x_3 = x_2 - \frac{f(x_2}{f'(x_2}$$
$$x_3 = \frac{1}{2} - \frac{ln(3/2) - 1/2}{2/3}$$

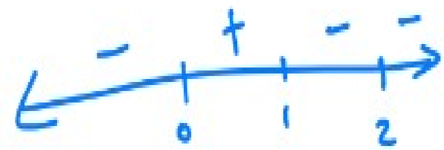
And our final answer is

$$1/2 - 3/2(ln(3/2) - 1/2) = 1/2 + 3/4 - 3/2ln(3/2)$$

$$\boxed{= \frac{5}{4} - \frac{3}{2}ln(3/2)}$$

2. Express $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sin(3+in^{-1})}{n}$ as a definite integral. Do not calculate the numerical value of that integral.
 Solution.
 $a = 3, b = 4 \Rightarrow dx = \Delta x = \frac{b-a}{n} = \frac{4-3}{n} = \frac{1}{n}$
 $x_i = 3 + \frac{1}{n}$
 $f(x) = \sin(x)$ thus the integral is $\boxed{\int_3^4 \sin(x)dx}$

3. Let $f(x) = (x^2 + x + 1)e^{-x}$. Find the critical points, maximum, minimum of f over $[0, 2]$. State where these values are attained.
Solution.



$f'(x) = (2x + 1)e^{-x} - e^{-x}(x^2 + x + 1) = 0$
$= e^{-x}(2x + 1 - (x^2 + x + 1)) = 0$
$= e^{-}x(-x^2 + x) = e^{-}xx(-x + 1)$
$x = 0, 1$
At $f(0) = 1$
$f(2) = (4 + 2 + 1)e^{-2} = \frac{7}{e^{-2}} < 1$
$f(1) = \frac{3}{e}$
Thus the minimum is at x = 2 and the maximum is at x = 1

4. Take the derivative of $f(x) = \frac{\ln(x+2)}{x^2-4x-3}$.
Solution.
Use Quotient Rule:

$$\boxed{f'(x) = \frac{\frac{x^2-4x-3}{x+2} - \ln(x+2)(2x-4)}{(x^2-4x-3)^2}}$$